arXiv:hep-th/0206053 v4 18 Apr 2003

Time-independant stochastic quantization, DS equations, and infrared critical exponents in QCD

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We derive the equations of time-independent stochastic quantization, without reference to an unphysical 5th time, from the principle of gauge equivalence. It asserts that probability distributions P that give the same expectation values for gauge-invariant observables $\langle W \rangle = \int dA \ W \ P$ are physically indistiguishable. This method escapes the Gribov critique. We derive an exact system of equations that closely resembles the Dyson-Schwinger equations of Faddeev-Popov theory. The system is truncated, and solved non-perturbatively, by means of a power law Ansatz, for the critical exponents that characterize the asymptotic form at $k \approx 0$ of the gluon propagator in Landau gauge. For the tranverse and longitudinal parts, we find respectively $D^T \sim (k^2)^{-1-\alpha_T} \approx (k^2)^{0.043}$, suppressed and in fact vanishing, though weakly, and $D^L \sim a \ (k^2)^{-1-\alpha_L} \approx a \ (k^2)^{-1.521}$, enhanced, with $\alpha_T = -2\alpha_L$. Although the longitudinal part vanishes with the gauge parameter a in the Landau-gauge limit, $a \to 0$, there are vertices of order a^{-1} so, counter-intuitively, the longitudinal part of the gluon propagator does contribute in internal lines in Landau gauge, replacing the ghost that occurs in Faddeev-Popov theory. We compare our results with the corresponding results in Faddeev-Popov theory.

1. Introduction

1.1. Some recent developments in non-perturbative QCD

The problem of the strong interaction presents an exciting challenge. One would like to understand how and why QCD describes a world of color-neutral hadrons with a mass gap, even though it appears perturbatively to be a theory of unconfined and massless gluons and quarks. Clearly an understanding of non-Abelian gauge theory at the non-perturbative level is required. Happily, there has recently developed a convergence of results by different methods: (i) non-perturbative solutions of the truncated Dyson-Schwinger (DS) equations in Faddeev-Popov theory, (ii) numerical evaluation of gauge-fixed, lattice QCD propagators, and (iii) exact analytic results. The agreement between these very different methods almost 5 decades after the appearance of the original article of Yang and Mills [1], would indicate that by (ii) we are beginning to get reliable values of the gluon propagator in the unbroken phase, and by (i) an understanding of the mechanism that determines it. This motivates the present investigation in which we derive the DS equations of time-independent stochastic quantization and solve them by truncation and a power-law Ansatz for the gluon propagator in the asymptotic, low-momentum régime. In accordance with earlier results by methods (i), (ii), and (iii), we find that, compared to the free propagator $1/k^2$, *the would-be physical, transverse component of the gluon propagator is short range, while the unphsical, longitudinal component is long range.*

As concerns (i), solutions of the DS equations, the decisive step was taken in [2], where a solution of the truncated DS equations in Faddeev-Popov quantization in Landau gauge was obtained for which *the transverse gluon propagator is short range, while the ghost propagator is long range.* These properties were confirmed in subsequent DS calculations, using a variety of approximations for the vertex [3], [4], [5], and [6]. More recent calculations extend the asymptotic infrared and ultraviolet solutions to finite momentum k, without angular approximation [7], [8]. All these calculations[1] give a transverse gluon propagator in Landau gauge $D^T(k)$ that is highly suppressed in the infrared compared to the free massless propagator $1/k^2$, and that in fact *vanishes* $\lim_{k \to 0} D^T(k) = 0$, at $k = 0$, in some cases weakly, like a small positive power of k. Indeed, according to the present calculation it vanishes like $(k^2)^{0.043}$. A review of DS equations in QCD may be

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[1] Stingl [9] had earlier obtained a solution of the DS equation with the property that $D^T(k)$ vanishes at $k = 0$, without however including the ghost loop, whereas the ghost loop gives the dominant contribution in the infrared region in the recent solutions.

found in [10]. In the present work we shall discover a close conection between the ghost propagator in Faddeev-Popov theory and the longitudinal part of the gluon propagator in time-independent stochastic quantization.

Concerning (ii), numerical studies, it is striking that an accumulation of numerical evaluations of the gluon propagator in Landau gauge also show qualitive suppression of the gluon propagator at low momentum, both in 3-dimensions on relatively large lattices, [11], [12], [13], and in 4 dimensions, [14], [15], [16], [17]. Suppression of the gluon propagator and enhancement of the ghost propagator at low momentum has been reported by [18], [19], and [20]. Similar numerical results were obtained in Coulomb gauge, where an extrapolation to infinite lattice volume of the 3-dimensionally transverse, would-be physical, equal-time gluon propagator $D_{ij}(\vec{k})$ was consistent with its vanishing at $\vec{k} = 0$, [21]. In QCD in the Coulomb gauge, the instantaneous Coulomb propagator, D_{44}, is closely related to the ghost or Faddeev-Popov propagator, and is a strong candidate for a confining potential. Significantly, D_{44} was found to be long range [21].

A recent numerical calculation in the Landau gauge, [16], reports a finite value of $D^T(k)$ at $k = 0$. This is strongly suppressed compared to $1/k^2$, and suffices to exclude a free massless gluon. It might be thought that the finite value of $D^T(0)$ reported in [16] contradicts the zero value, $D^T(0) = 0$, found here. However it is difficult to distinguish numerically between a finite value at $k = 0$ and one that vanishes weakly, like $(k^2)^\gamma$, with a small value for the infrared anomalous dimension such as $\gamma = 0.043$ found here. For this function is almost constant down to very low k, and then veers toward zero with an infinite slope. Moreover a numerical determination of the continuum propagator at $k = 0$ requires an extrapolation to infinite lattice volume. To establish a discrepancy it would be necessary to take γ as a fitting parameter, and determine the numerical uncertainty in this quantity after extrapolation to infinite lattice volume, and this has not been done. Present numerical and analytic results are not inconsistent, within the considerable uncertainty of the numerical extrapolation to infinite lattice volume, and both agree that there is strong suppression compared to $1/k^2$.

Infrared suppression of the gluon propagator $D(k)$ and enhancement of the ghost propagator $G(k)$ in Landau gauge was first found by Gribov, using avowedly rough approximations [22]. He obtained the formulas, $D(k) = k^2/[(k^2)^2 + M^4]$, and, in the infrared, $G(k) \sim 1/(k^2)^2$, by restricting the region of functional integration to the interior

of the Gribov horizon in order to avoid Gribov copies.[2] In Coulomb gauge he also obtained a long-range Coulomb potential. Concerning (iii), exact analytic results, it was subsequently found, [25] and [26], that restriction to the interior of the Gribov horizon, enforced by a *horizon condition,* yields at $k = 0$, both the vanishing of the gluon propagator $\lim_{k \to 0} D(k) = 0$ in Landau and Coulomb gauge, and the enhancement of the ghost propagator $\lim_{k \to 0} k^2 G(k) = \infty$.[3]

It was at first surprising that the solution of the DS equations obtained in [2], [3], [4], and [5] agreed with these exact results that are a consequence of cutting off the functional integral at the Gribov horizon, for this condition was not imposed in solving the DS equations. However it was subsequently pointed out [6] that the DS equations in Faddeev-Popov theory depend only on the integrand, and the fact that the integral of a derivative vanishes *provided only that the integrand vanishes on the boundary.* The key point is that *the integrand does vanish on the Gribov horizon* for the Faddeev-Popov determinent, $\det[-D(A) \cdot \partial]$, vanishes there (as explained in footnote 2). Thus Gribov's prescription to cut off the functional integral at the (first) Gribov horizon, is not a constraint that changes the DS equations, but rather it resolves an ambiguity in the solution of these equations [6]. The cut-off at the first Gribov horizon assures that both the gluon and ghost Euclidean propagators are positive, which is a property of the solutions obtained for the truncated DS equations. Moreover the solution of the DS equations in Faddeev-Popov theory with a cut-off at the Gribov horizon is the only one for which a comparison with numerical gauge fixing to the lattice Landau gauge is (approximately) justified. For as explained in footnote

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[2] We remind the reader that numerical gauge-fixing to the Landau gauge is achieved by minimizing, with respect to local gauge transformations $g(x)$, a lattice analog of $F_A(g) = \int d^4 x |^g A|^2$. At any minimum, this functional is (a) stationary, and (b) the matrix of second derivatives is non-negative. These conditions correspond to (a) the Landau gauge condition $\partial \cdot A = 0$, and (b) the positivity of the Faddeev-Popov operator $-D(A) \cdot \partial$ which, moreover is symmetric $-D(A) \cdot \partial = -\partial \cdot D(A)$, for $\partial \cdot A = 0$. Condition (b) defines the Gribov region, so numerical studies of the Landau gauge automatically select configurations within the Gribov region. Positivity of $-\partial \cdot D(A)$ means that all its eigenvalues λ_n are positive, and the boundary of the Gribov region, known as the (first) Gribov horizon, is where the first (non-trivial) eigenvalue vanishes. Thus the Faddeev-Popov determinent, $\det[-D(A) \cdot \partial] = \prod_n \lambda_n$, which is the product of the eigenvalues, is positive inside the Gribov horizon and vanishes on it. These considerations do not apply to numerical gauge fixing to the Laplacian gauge [23], [24].

[3] It is noteworthy that the confinement criterion of Kugo and Ojima [27] and [28] also entails $\lim_{k \to 0} k^2 G(k) = \infty$.

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2, numerical gauge fixing to the Landau gauge automatically produces a configuration that lies inside the Gribov horizon. Thus a consistent picture emerges of the gluon and ghost propagators in QCD using the different methods (i), (ii) and (iii).

1.2. Diffulties of Faddeev-Popov method at non-perturbative level

The DS calculations [2] – [5] rely on Faddeev-Popov theory which however is subject to the well-known critiques of Gribov [22] and Singer [29]. At the perturbative level, Faddeev-Popov theory is unexceptionable, and elegant BRST proofs are available of perturbative renormalizability and perturbative unitarity [30]. In lattice gauge theory however the BRST method fails because the total number of Gribov copies is even, but they contribute with opposite signs, leading to an exact cancellation [31], [32]. In continuum gauge theory, the Faddeev-Popov-BRST method may nevertheless be formally correct at the non-perturbative level without a cut-off at the Gribov horizon, if ones sums over all signed Gribov copies [33], [34]. However even if this is true, it would imply large cancellations between copies, that may amplify the error of an approximate non-perturbative calculation, and even the Euclidean gluon propagator $D(k)$ is not necessarily positive. Alternatively, one may choose the solution of the DS equations in Faddeev-Popov theory that corresponds to a cut-off at the first Gribov horizon, which indeed is our interpretation of the solutions of [2] — [8]. Hopefully, this is an excellent approximation. But it remains an *ad hoc* prescription that is not correct in principle because of the existence of Gribov copies inside the Gribov horizon [35] and [36].

Wilson's lattice gauge theory provides a quantization that is both theoretically sound and well suited to numerical simulation. It also provides a simple analytic model of confinement in QCD by giving an area law for Wilson loops in the strong-coupling limit. A striking feature of lattice gauge theory is that both the numerical simulations and the strong-coupling expansion are manifestly gauge invariant. This manifest gauge invariance provides a paradigm for continuing efforts to understand confinement in QCD. Nevertheless it may be worthwile to pursue other approaches. The vexing problem of bound states in quantum field theory is particularly urgent in QCD where confinement causes all physical particles to be bound states of the fundamental quark and gluon constituents. In this regard it is noteworthy that even the simplest of all bound-state problems, the Hydrogen atom, is not easily solved in a gauge-invariant formulation.

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1.3. Review of stochastic quantization

In order to avoid the difficulties just mentioned of the Faddeev-Popov method, we turn to stochastic quantization of gauge fields for, as we shall see, this method provides a correct continuum quantization at the non-perturbative level. Stochastic quantization has been developed by a number of authors [37], [38], who have expressed the solution as a functional integral [39], and demonstrated the renormalizability of this approach [40], [41]. A systematic development is presented in [42], [43], [44], [45], [46], [47], reviewed in [48], that includes the 4-and 5-dimensional Dyson-Schwinger equation for the quantum effective action, an extension of the method to gravity, and gauge-invariant regularization by smoothing in the 5th time. Renormalizability has also been established by an elaboration of BRST techniques [49], [50]. Stochastic quantization may be and has been exactly simulated numerically including on rather large lattices, of volume $(48)^4$, [51], [52], [53], [54], [55]. This suggests the possibility of a promising interplay of DS and numerical methods.

In its original formulation [37], stochastic quantization relies on the observation that the formal Euclidean proability distribution $P_0(A) = N \exp[-S_{\mathrm{YM}}(A)]$, with 4-dimensional Euclidean Yang-Mills action $S_{\mathrm{YM}}(A)$, is the equilibrium distribution of the stochastic process defined by the equation,

$$\frac{\partial P}{\partial t} = \int d^4x \, \frac{\delta}{\delta A_\mu^a(x)} \left(\frac{\delta P}{\delta A_\mu^a(x)} + \frac{\delta S_{\mathrm{YM}}}{\delta A_\mu^a(x)} P \right). \tag{1.1}$$

Indeed it is obvious that $P_0(A)$ is a time-independent solution of this equation. Here t is an artificial 5th time that is a continuum analog of the number of sweeps in a Monte Carlo simulation of the Euclidean theory defined by the action $S_{\mathrm{YM}}(A)$. As explained in sec. 3, this equation has the form of the diffusion equation with "drift force" $-\frac{\delta S_{\mathrm{YM}}}{\delta A_\mu^a(x)}$, and is known as the Fokker-Planck equation. The same stochastic process may equivalently be represented by the Langevin equation

$$\frac{\partial A_\mu^a}{\partial t} = -\frac{\delta S_{\mathrm{YM}}}{\delta A_\mu^a} + \eta_\mu^a, \tag{1.2}$$

where $A_\mu^a = A_\mu^a(x,t)$ depends on the artificial 5th time, and corresponds in a Monte-Carlo simulation to the configuration on the lattice with points x_μ, with $\mu = 1, ...4$ at sweep t. Here $\eta_\mu^a = \eta_\mu^a(x,t)$ is Gaussian white noise defined by $\langle \eta_\mu^a(x,t) \rangle = 0$ and $\langle \eta_\nu^b(x,t)\eta_\mu^a(y,t') \rangle = 2\delta(x-y)\delta_{\mu\nu}\delta^{ab}\delta(t-t') \rangle$. If $N\exp[-S_{\mathrm{YM}}(A)]$ were a normalizable probability distribution — which it is not — every normalized solution to (1.1) would relax to it as equilibrium

distribution. However the process defined by (1.1) or (1.2) does not provide a restoring force in gauge orbit directions, so probability escapes to infinity along the gauge orbits, and as a result $P(A, t)$ does not relax to a well-defined limitiing distribution $\lim_{t\to\infty} P(A, t) \neq N\exp[-S_{\text{YM}}(A)]$. Nevertheless, according to [37], expectation values $\langle \mathcal{O}(A) \rangle_t$ of gauge-invariant quantities $\mathcal{O}(A)$ calculated at fixed but finite time t according to either of the above equations do relax to the desired Euclidean expectation value, $\langle \mathcal{O} \rangle = \lim_{t\to\infty} \langle \mathcal{O} \rangle_t$.

Unfortunately the renormalization program cannot be carried out in this scheme as stated, because that requires that gauge-non-invariant correlators also be well defined. A remedy is provided by the observation [38] that the Langevin equation may be modified by the addition of an infinitesimal gauge transformation, $D_\mu^{ac} v^c = (\partial_\mu \delta^{ac} + f^{abc} A_\mu^b) v^c$,

$$\frac{\partial A_\mu^a}{\partial t} = -\frac{\delta S}{\delta A_\mu^a} + D_\mu^{ac} v^c + \eta_\mu^a. \tag{1.3}$$

Clearly this cannot alter the expecation-value of gauge-invariant quantities. Symmetry and power-counting arguments determine $v^a = a^{-1} \partial_\lambda A_\lambda^a = a^{-1} \partial \cdot A^a$, where a is a free parameter. For $a > 0$, the new term, that is tangent to the gauge orbit, provides a restoring force along gauge orbit directions, so gauge-non-invariant correlators also exist.[4] The new scheme is renormalizable. Only a harmless gauge-transformation has been introduced, so the Gribov problem of globally correct gauge-fixing is by-passed, and a continuum quantization of gauge fields that is correct at the non-perturbative level has been achieved.

The modified Langevin equation is equivalent to the modified Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \int d^4x \frac{\delta}{\delta A_\mu^a(x)} \left(\frac{\delta P}{\delta A_\mu^a(x)} - K_\mu^a(x) P \right), \tag{1.4}$$

where the "drift force" now includes the infinitesimal gauge transformation [38],

$$K_\mu^a(x) \equiv -\frac{\delta S_{\text{YM}}}{\delta A_\mu^a(x)} + a^{-1} D_\mu^{ac} \partial \cdot A^c(x). \tag{1.5}$$

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[4] To establish that the new force is globally restoring, we note that the hilbert norm of A is decreasing under the flow defined by the new force alone, $\dot{A}_\mu = a^{-1} D_\mu \partial \cdot A$. We have $\partial \|A\|^2 / \partial t = 2(A_\mu, \dot{A}_\mu) = 2a^{-1}(A_\mu, D_\mu \partial \cdot A) = 2a^{-1}(A_\mu, \partial_\mu \partial \cdot A) = -2a^{-1}\|\partial \cdot A\|^2 \leq 0$. This also shows that the region of equilibrium under this force is the set of transverse configurations, $\partial \cdot A = 0$. Similarly, from $\partial \|\partial \cdot A\|^2 / \partial t = 2(\partial \cdot A, \partial \cdot \dot{A}) = 2a^{-1}(\partial \cdot A, \partial \cdot D \partial \cdot A)$ it follows that this equilibrium is *stable* inside the Gribov horizon, where $-\partial \cdot D$ is a positive operator, and *unstable* outside it.

The additional "force" is not conservative, and cannot be written, like the first term, as the gradient of some 4-dimensional gauge-fixing action, $a^{-1}D_\mu^{ac}\partial \cdot A^c(x) \neq -\frac{\delta S_{\text{gf}}}{\delta A_\mu^a(x)}$. With this term, the normalized solutions $P(A,t)$ to (1.4) do relax to an equilibrium distribution $\lim_{t\to\infty} P(A,t) = P(A)$, and Euclidean expectation values are given by the 4-dimensional functional integral, $\langle \mathcal{O}\rangle = \int dA \ \mathcal{O}(A)P(A)$. Although we cannot write $P(A)$ explicitly because the force is not conservative, we do know that it is the normalized solution of the time-independent Fokker-Planck equation

$$H_{\text{FP}}P \equiv \int d^4x \frac{\delta}{\delta A_\mu^a(x)}\Big(-\frac{\delta P}{\delta A_\mu^a(x)} + K_\mu^a P\Big) = 0. \qquad (1.6)$$

This equation defines what we call "time-independent stochastic quantization", and H_{FP} is called the "Fokker-Planck hamiltonian". The solution $P(A)$ of this equation provides a satisfactory non-perturbative quantization of gauge fields.

[To avoid possible confusion of terminology, we note that stochastic quantization, whether in the time-dependent or time-independent formulation, — where "time" is the artificial 5th time — increases the number of dimensions by one as compared to the corresponding standard Faddeev-Popov formulation of gauge field theory. Thus the solution of the time-dependent Fokker-Planck equation (1.4) can be usefully represented [39] as a functional integral with a *local 5-dimensional* action $I = \int dt d^4x \mathcal{L}_5$, whereas in Faddeev-Popov theory, expectation values may be calculated by a functional integral with a *local 4-dimensional* action $S = \int d^4x[(1/4)F_{\mu\nu}^2 + ...]$. Likewise the Fokker-Planck "hamiltonian" H_{FP} determines, by the time-independent Fokker-Planck equation $H_{\text{FP}}P = 0$, a Euclidean probability distribution $P(A)$ whose argument is a field $A(x)$ that is a function in *4-dimensional* space-time with points x_μ, $\mu = 1, ...4$. By comparison the quantum mechanical hamiltonian H_{QM} in ordinary quantum field theory determines, by the time-independent Schrödinger equation $H_{\text{QM}}\Psi = E\Psi$, a wave-functional $\Psi(A)$, whose argument is a field $A(\vec{x})$ that is a function in ordinary *3-space* $\vec{x} = (x_1, x_2, x_3)$. Thus H_{FP} is not a quantum mechanical hamiltonian at all, but rather, it claims the name "hamiltonian" as the generator of time translations in the time-dependent Fokker-Planck equation (1.4), where the "time" is the artificial 5th time. Unlike the quantum-mechanical hamiltonian formulation, time-independent stochastic quantization is 4-dimensionally Lorentz (Euclidean) covariant.]

Despite the development of stochastic quantization in [37] — [50] it has apparently not so far been used for non-perturbative calculations in QCD, apart from [6].[5] This may possibly be due to the complication caused by the extra "time" variable. Although the time-dependent formulation allows an elegant representation, with a local 5-dimensional action, it has the complication in practice that the gluon propagator depends on two invariants $D(k^2, \omega)$ instead of only one $D(k^2)$. This prevents a simple power Ansatz for the infrared behavior $1/(k^2)^{1+\alpha}$ that allows one to determine the infrared behavior of the 4-dimensional theory self-consistently. For this reason we turn to time-independent stochastic quantization, where the correlators have the same number of invariants as in Faddeev-Popov theory.

1.4. Outline of the present article

We shall not use the 5-dimensional formulation here, but only the 4-dimensional, time-independent Fokker-Planck equation (1.6). The solution $P(A)$ to this equation cannot be represented as a functional integral over a local 4-dimensional action. Nor shall we attempt to construct an explicit solution to (1.6). Our strategy instead will be to convert it into a system of tractable DS equations for the correlators.

As a first step, we convert (1.6) into the DS equation, (6.3) below, for the quantum effective action Γ. The DS equation for Γ appears relatively complicated, with a *second-order* structure inherited from the second-order operator in (1.6). The main methodological innovation of the present approach is that the second-order equation for Γ is replaced, in secs. 6 and 7, by the much simpler DS equation (6.6) for a quantity, $\mathcal{Q}_\mu^a(x)$, that we call "the quantum effective drift force". Indeed the new equation $\mathcal{Q}_\mu^a(x) = K_\mu^a(x) +$ (loop integrals), where $K_\mu^a(x) = -\frac{\delta S_{\mathrm{YM}}}{\delta A_\mu^a(x)} + a^{-1}(D_\mu \partial \cdot A)^c$, has the same structure as the *first-order* DS equation for Γ in Faddeev-Popov theory, $\frac{\delta \Gamma}{\delta A_\mu^a(x)} = \frac{\delta S}{\delta A_\mu^a(x)} +$ (loop integrals). In both of these equations, the leading term may be interpreted as a drift force and, most helpfully for the renormalization program, it is local in $A(x)$.

In the present work we give an improved treatment, as compared to [6], of the longitudinal degrees of freedom in the Landau-gauge limit $a \to 0$. In that work we integrated out the longitudinal degrees of freedom in the Landau-gauge limit $a \to 0$. This gave a time-independent Fokker-Planck equation for the transverse degrees of freedom only, with

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[5] The equations of stochastic quantization have however been applied to dissipative problems in QCD, where t is the physical time, and x physical 3-space, [56].

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an effective drift force that was however non-polynomial and non-local.[6] By contrast, in the present work, the difficulty of a non-polynomial drift force is avoided by retaining the longitudinal degrees of freedom. Of course the longitudinal part of the propagator vanishes with the gauge-parameter a in the Landau-gauge limit $\lim \to 0$. However the drift force (1.5) gives a vertex that diverges like $1/a$ and so, counter-intuitively, the longitudinal part of the propagator in the Landau gauge limit gives a finite contribution in internal loops, somewhat like the ghost in Faddeev-Popov theory.

We shall be satisfied here to calculate only the infrared asymptotic form of the propagator, because that is where the challenging, non-perturbative confining phenomena manifest themselves. At high momentum, QCD is perturbative, and it has been verified to one-loop order by various methods [57]and [46], including the background field method [58], that stochastic quantization yields the standard β-function. We leave for another occasion a numerical calculation which would be necessary to connect the high- and low-momentum limits.

Since we use only the time-independent formulation here, we present, in secs. 2 and 3, a new derivation of eq. (1.6) that does not refer to the unphysical 5th time. At the end of sec. 3 the Minkowskian form of time-independent stochastic quantization is presented. [Some readers may prefer to go directly to sec. 4, which begins with (1.6).] The new derivation is more powerful, and yields new results, in particular, the Ward identity of Appendix C, and the proof in Appendix A that the kernel of the Fokker-Planck hamiltonian for quarks depends on gauge parameters only. We shall derive it from the obvious principle of *gauge equivalence* which asserts that probability distributions $P(A)$ that give the same expectation values for gauge-invariant observables $\langle W \rangle = \int dA \, W(A) \, P(A)$ are physically indistiguishable. We show that time-independent stochastic quantization provides a class of positive, normalized proability distributions $P(A, a)$, parametrized by a gauge parameter a that are gauge equivalent $P(A, a_1) \sim P(A, a_2)$, and that includes includes the Yang-Mills distribution $N \exp(-S_{\mathrm{YM}})$ as a limiting case. This method of quantization of gauge fields, in which the unphysical degrees of freedom are retained but controlled, is closely related to the physics of our solution of the DS equations. Indeed we find that the physical degrees of freedom are short range, whereas the unphysical degrees of freedom are not only present but of long range. In Appendix A, we extend the method to include quarks, and

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[6] This was in turn decomposed into a conservative force that reproduced the Faddeev-Popov determinent, plus a second term that was neglected in the solution found in [6].

in Appendix B, to lattice gauge theory. In Appendix C, we derive a Ward identity that controls the divergences of the theory.

In sec. 8 we derive the explicit form of the DS equation for the gluon propagator. In secs. 9 – 11 we adopt a simple truncation scheme, and by means of a power-law Ansatz we solve for the infrared critical exponents that characterize the gluon propagator in Landau gauge asymptotically, at low momentum. The transverse part of the gluon propagator is short range, and the longitudinal part long range. In the concluding section we compare our results with calculations in Faddeev-Popov theory, and we interpret their qualitative features in a confinement scenario. We also suggest some challenging open problems, and possibilities for comparison with numerical simulation in lattice gauge theory.

2. Gauge equivalence

We first consider Euclidean gauge theory and later the Minkowskian case. Non-abelian gauge theories are described by the Yang-Mills action $S_{\mathrm{YM}}(A) = (1/4) \int d^4x (F_{\mu\nu}^a)^2$, where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$. The Euclidean quantum field theory is formally defined by the probability distribution $P_{\mathrm{YM}}(A) = N \exp[-S_{\mathrm{YM}}(A)]$, and by the expectation-values $\langle W \rangle = \int dA \, W(A) \, P_{\mathrm{YM}}(A)$, normalized so $\langle 1 \rangle = 1$. The challenge of quantizing a non-abelian gauge theory is that $P_{\mathrm{YM}}(A)$ is not really normalizable because of the infinite volume of the local gauge group.

The challenge would be hopeless, but for the fact that we are interested only in observables that are invariant under local gauge transformations, $W(^gA) = W(A)$, for all $g(x)$, where $^gA_\mu = g^{-1}A_\mu g + g^{-1}\partial_\mu g$. This suggests the notion of *gauge-equivalent* probability distributions. Two probability distributions are gauge equivalent, $P_1(A) \sim P_2(A)$, if and only if $\langle W \rangle_1 = \langle W \rangle_2$, for all gauge-invariant observables W, where $\langle W \rangle_i = \int dA \, W(A) \, P_i(A)$. Gauge equivalence of probability distributions is dual to gauge invariance of observables. Distributions that are gauge-equivalent are physically indistinguishable. Our solution to the quantization problem will be to replace the formal probability distribution $N \exp[-S_{\mathrm{YM}}(A)]$, by a normalizable distribution that is gauge-equivalent to it. More precisely we shall exhibit a class of gauge-equivalent normalized distributions that includes $N \exp[-S_{\mathrm{YM}}(A)]$ as a limiting case.

3. A machine that makes gauge-equivalent probability distributions

The construction of gauge-equivalent probability distributions relies on an equation that has the same form as the time-independent Fokker-Planck equation that is used to describe diffusion in the presence of a drift force. In this section, for simplicity, we deal with continuum gauge fields, or gluodynamics, only. The extension to quarks is given in Appendix A, and to lattice gauge theory in Appendix B.

In order to simplify the appearance of various equations, we shall, as convenient, use the index notation A_x, instead of $A_\mu^a(x)$, where the subscript x represents the triplet x, μ, a. We use discrete notation and the summation convention on the new index so, for example, $\frac{\partial J_x}{\partial A_x}$ replaces $\int d^4x \, \frac{\delta J_\mu^a(x)}{\delta A_\mu^a(x)}$.

Let $P(A)$ be a positive, $P(A) > 0$, normalized, $\int dA \, P(A) = 1$, probability distribution or concentration. In simple diffusion theory there is associated with this distribution a current,

$$J_x = -\hbar \frac{\partial P}{\partial A_x} + K_x P, \tag{3.1}$$

that is composed of a diffusive term, $-\hbar \frac{\partial P}{\partial A_x}$, proportional to the gradient of the concentration, with diffusion constant $\hbar$, and a drift term, $K_x P$. Here K_x is the drift force, as in Ohm's law with unit conductivity. We have introduced $\hbar$ for future convenience for a loop expansion which is an expansion in powers of $\hbar$. Conservation of probability is expressed by the equation of continuity $\frac{\partial P}{\partial t} = -\frac{\partial J_x}{\partial A_x}$. The analogy of interest to us here is associated with the time-independent situation only.[7] In this case the current is divergenceless

$$\frac{\partial J_x}{\partial A_x} = 0, \tag{3.2}$$

which reads

$$H_{\mathrm{FP}}P \equiv \frac{\partial}{\partial A_x} \left(-\hbar \frac{\partial}{\partial A_x} + K_x\right) P = 0. \tag{3.3}$$

This is the time-independent diffusion equation with drift force K_x. We call the linear operator defined here the Fokker-Planck "hamiltonian", although H_{FP} is not hermitian,

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[7] Stochastic quantization [37], including a drift force tangent to the gauge orbit [38], has traditionally been based on the time-dependent Fokker-Planck equation $\frac{\partial P}{\partial t} = -H_{\mathrm{FP}}P$, and relied on relaxation of the stochastic process to an equilibrium distribution that satisfies $H_{\mathrm{FP}}P = 0$. Here t is an additional, unphysical time variable that corresponds to computer time in a Monte Carlo simulation. By contrast, in the present article, the quantization of the non-Abelian gauge field follows from the geometrical principle of gauge equivalence, from which we derive the time-independent equation $H_{\mathrm{FP}}P = 0$ directly, without reference to the additional time variable.

as it would be in quantum mechanics, and it certainly is not the quantum mechanical hamiltonian of the gauge field.

We must be sure to choose a drift force that is restoring, so this equation determines a positive normalized distribution $P(A)$. If the drift force were conservative $K_x = -\frac{\partial S_{\mathrm{YM}}}{\partial A_x}$, then the normalized solution would be $P(A) = N \exp(-S_{\mathrm{YM}})$. Gauge invariance of the Yang-Mills action, $D_\mu^a \frac{\delta S_{\mathrm{YM}}}{\delta A_\mu^a(x)} = 0$, means however that the conservative drift force $-\frac{\delta S_{\mathrm{YM}}}{\delta A_\mu(x)}$ provides no restoring force in gauge-orbit directions. This is remedied by introducing an additional force $K_{\mathrm{gt},\mu}^a(x) = D_\mu^{ac} v^c$ that is an infinitesimal gauge transformation, so the drift force is made of a conservative piece and a piece that is tangent to the gauge orbit,

$$
\begin{aligned}
K_\mu^a(x) &= -\frac{\delta S_{\mathrm{YM}}}{\delta A_\mu^a(x)} + K_{\mathrm{gt},\mu}^a(x) \\
&= -\frac{\delta S_{\mathrm{YM}}}{\delta A_\mu^a(x)} + D_\mu^{ac} v^c.
\end{aligned}
\tag{3.4}
$$

Geometrically, the drift force is a vector field or flow, and it is intuitively clear that a flow that is tangent to the gauge orbit has no effect on gauge-invariant observables. We will not fail to choose $v^a(x; A)$ so that $D_\mu v$ is a restoring force, to insure that (3.3) possesses a positive, normalized solution. Apart from this restoring property, $v^a(x; A)$ may in principle be an arbitrary functional of A. The time-independent Fokker-Planck equation reads explicitly

$$
H_{\mathrm{FP}} P \equiv \int d^4 x \, \frac{\delta}{\delta A_\mu^a(x)} \left[-\hbar \frac{\delta P}{\delta A_\mu^a(x)} + \left(-\frac{\delta S_{\mathrm{YM}}}{\delta A_\mu^a(x)} + D_\mu^{ac} v^c \right) P \right] = 0.
\tag{3.5}
$$

This equation is a machine that produces normalized probability distributions $P_v(A)$ that are gauge equivalent to $N \exp(-S_{\mathrm{YM}})$.

We now prove the basic result. *Positive, normalized solutions of the diffusion equation (3.5) for different v are gauge equivalent $P_v \sim P_{v'}$, and include $N \exp[-S_{\mathrm{YM}}(A)]$ as a limiting case.* Our solution to the problem of quantizing a gauge field is to use any one of the $P_v(A)$ to calculate expectation-values of gauge-invariant observables. We consider observables that are invariant under infinitesimal local gauge transformations, namely that satisfy $G^a(x) W = 0$. Here $G^a(x) \equiv -D_\mu^{ac} \frac{\delta}{\delta A_\mu^c(x)}$, is the generator of an infinitesimal gauge transformation, with local Lie algebra, $[G^a(x), G^b(y)] = \delta(x-y) f^{abc} G^c(x)$, and $(D_\mu X)^a \equiv \partial_\mu X^a + g f^{abc} A_\mu^b X^c$ is the gauge-covariant derivative in the adjoint representation.

The proof relies upon the decomposition of H_{FP},

$$H_{\text{FP}} = H_{\text{inv}} - (v, G)^{\dagger}$$

$$H_{\text{inv}} \equiv \int d^4x \, \frac{\delta}{\delta A_\mu^a(x)} \left[- \hbar \frac{\delta}{\delta A_\mu^a(x)} - \frac{\delta S_{\text{YM}}}{\delta A_\mu^a(x)} \right]$$

$$(v, G) \equiv - \int d^4x \, v^a D_\mu^{ac} \frac{\delta}{\delta A_\mu^c(x)} = \int d^4x \, (D_\mu v)^a \frac{\delta}{\delta A_\mu^a(x)} \tag{3.6}$$

$$- (v, G)^{\dagger} = (G, v) = \int d^4x \, \frac{\delta}{\delta A_\mu^a(x)} (D_\mu v)^a$$

where $\dagger$ is the adjoint with respect to the inner product defined by $\int dA$, and (v, G) is the generator of the local gauge transformation $v^a(x)$. Note that H_{inv} is a gauge-invariant operator, $[G^a(x), H_{\text{inv}}] = 0$, that has $\exp(-S_{\text{YM}})$ as a null vector, $H_{\text{inv}} \exp(-S_{\text{YM}}) = 0$. Let $P(A)$ be the normalized solution of $H_{\text{FP}} P = 0$ for given v. It is sufficient to show that $\langle W \rangle = \int dA \, W(A) P(A)$ is independent of $v^a(x)$ for gauge-invariant observables W. Let $\delta v^a(x)$ be an arbitrary infinitesimal variation of $v^a(x)$. The corresponding change in $P(A)$ satisfies $\delta H_{\text{FP}} P + H_{\text{FP}} \delta P = 0$, where $\delta H_{\text{FP}} = (G, \delta v)$, so

$$\delta P = - H_{\text{FP}}^{-1} \, \delta H_{\text{FP}} \, P. \tag{3.7}$$

Note that $\delta H_{\text{FP}} \, P$ has the form of a divergence, so it is orthogonal to the null space of H_{FP}. This change in P induces the change in expectation value

$$\begin{aligned}
\delta \langle W \rangle &= \int dA \, \delta P \, W \\
&= - \int dA \, (H_{\text{FP}}^{-1} \, \delta H_{\text{FP}} \, P) \, W \\
&= - \int dA \, P \, [\delta H_{\text{FP}}^{\dagger} \, (H_{\text{FP}}^{\dagger})^{-1} \, W] \\
&= \int dA \, P \, [(\delta v, G) \, (H_{\text{FP}}^{\dagger})^{-1} \, W],
\end{aligned} \tag{3.8}$$

where $H_{\text{FP}}^{\dagger} = H_{\text{inv}}^{\dagger} - (v, G)$. It is sufficient to show that $\delta \langle W \rangle = 0$. The proof is almost immediate, but we must verify that the dependence of $v^a(x; A)$ and $\delta v^a(x; A)$ on A does not cause any problem. Recall that W is gauge invariant, $G^a(x) W = 0$, so we have $H_{\text{FP}}^{\dagger} W = H_{\text{inv}}^{\dagger} W$, which implies that $H_{\text{FP}}^{\dagger} W$ is gauge invariant,

$$G^a(x) \, H_{\text{FP}}^{\dagger} W = G^a(x) \, H_{\text{inv}}^{\dagger} W = 0.$$

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It follows by induction that $(H_{\text{FP}}^\dagger)^n W = (H_{\text{inv}}^\dagger)^n W$ is gauge invariant for any integer n, $G^a(x)\,(H_{\text{FP}}^\dagger)^n W = 0$, which implies that for any analytic function, $f(H_{\text{FP}}^\dagger)W = f(H_{\text{inv}}^\dagger)W$ is gauge invariant $G^a(x)\,f(H_{\text{FP}}^\dagger)\,W = 0$. This holds in particular for $f(z) = 1/z$, and we have $G^a(x)\,(H_{\text{FP}}^\dagger)^{-1}\,W = 0$. This implies that $\delta\langle W\rangle = 0$ as asserted. Note also that if $v = 0$, then the formal solution is $P = N\exp(-S_{\text{YM}})/\hbar$.

This proof does not rely on Faddeev-Popov gauge-fixing which would require a gauge choice that selects a single representative on each gauge orbit. The Gribov critique is by-passed, and Singer's theorem [29] does not apply. Gauge equivalence is a weaker condition than gauge fixing, but sufficient for physics. In the present approach we do not attempt to eliminate "unphysical" variables and keep only "physical" degrees of freedom. Rather we work in the full A-space, keeping all variables, but taming the gauge degrees of freedom by exploiting the freedom of gauge equivalence. It is the unphysical degrees of freedom that provide a long-range correlator, and a strong candidate for a confining potential.

Another way to obtain a gauge-equivalent probability distribution is by gauge transformation. If our class of gauge-equivalent probability distributions $P_v(A)$ is large enough, then it is possible to absorb an infinitesimal gauge transformation $\delta A_\mu = D_\mu\epsilon$ by an appropriate change δv of v, $P_v(A + D_\mu\epsilon) = P_{v+\delta v}(A)$. This is true and leads to a useful Ward identity that is derived in Appendix C.

There remains to choose v so it has a globally restoring property. An optimal way to do this is to require that the force $D_\mu v$, that is tangent to the gauge orbit, points along the direction of steepest descent, restricted to gauge-orbit directions, of a conveniently chosen functional. For the minimizing functional, we take the Hilbert norm-square, $F(A) = ||A||^2 = \int d^4x\,|A|^2$, and we consider a variation $\delta A_\mu = \eta D_\mu v$ that is tangent to the gauge orbit in the v-direction, where η is an infinitesimal parameter. We have

$$
\begin{aligned}
\delta F &= 2(A, \delta A) = 2\eta(A, Dv) = 2\eta \int d^4x\, A_\mu^a\,(\partial_\mu v^a + f^{abc}A_\mu^b v^c) \\
&= -2\eta \int d^4x\,\partial_\mu A_\mu^a\,v^a.
\end{aligned}
\tag{3.9}
$$

Thus the direction of steepest descent, restricted to gauge orbit directions, is given by

$$
v^a = a^{-1}\partial_\mu A_\mu^a,
\tag{3.10}
$$

where $a > 0$ is a positive constant. We shall take this optimal choice for v, so the total drift force that appears in the diffusion equation is given by[8]

$$
K_\lambda^a(x; A) = D_\mu^{ac}F_{\mu\lambda}^c + a^{-1}D_\lambda^{ac}\partial_\mu A_\mu^c.
\tag{3.11}
$$

---

[8] An alternative choice suitable for the Higgs phase was proposed in [50].

Here a is a dimensionless gauge parameter. This completes the specification of the time-independent stochastic quantization.

The drift force $a^{-1}D_\lambda \partial_\mu A_\mu$ tends to concentrate the probability distribution $P(A)$ close to its region of stable equilibrium, especially if a is small. Let us find the region of stable equilibrium. From (3.9) we see that $\delta F < 0$ unless A satisfies $\partial_\mu A_\mu = 0$. This defines the region of equilibrium, which may be stable or unstable. The region of (local) stable equilibrium is determined by the additional condition that the second variation be non-negative $\delta^2 F > 0$, for all variations δA tangent to the gauge orbit, namely $\delta A = D_\mu \epsilon$, for arbitrary $\epsilon^a(x)$. We have just found that the first variation is given by $\delta F = -2(\epsilon, \partial_\mu A_\mu)$. So we have, for the second variation, $\delta^2 F = -2(\epsilon, \partial_\mu \delta A_\mu) = -2(\epsilon, \partial_\mu D_\mu \epsilon)$. Thus the region of *stable* equilibrium is determined by the two conditions $\partial_\mu A_\mu = 0$ and $-\partial_\mu D_\mu(A) > 0$, namely transverse configurations A, for which the Faddeev-Popov operator $-\partial_\mu D_\mu(A)$ is positive. These two conditions define the Gribov region. We expect that in the limit $a \to 0$, both conditions will be satisfied. This is the Landau gauge, with probability restricted to the interior of the first Gribov horizon.

So far we have discussed Euclidean quantum field theory, which is characterized by elliptic differential operators. However the above considerations also apply to the Minkowski case. Here the formal weight is $Q(A) = N \exp[iS_{\rm YM}]$, where $S_{\rm YM} = (-1/4) \int d^4x\, F^{\mu\nu}F_{\mu\nu}$ is the Minkowskian Yang-Mills action, where indices are raised and lowered by the metric $g_{\lambda\mu} = g^{\lambda\mu} = {\rm diag}(1,1,1,-1)$. Expectation-values of gauge-invariant time-ordered observables, are given by the Feynman path integral $\langle W \rangle = \int dA\, W(A)\, Q(A)$, with $\langle 1 \rangle = 1$. Instead of eq. (3.5), we take gauge-equivalent configurations that are solution of the equation

$$H_{\rm M}Q = 0, \tag{3.12}$$

where $H_{\rm M}$ is the corresponding Minkowskian "hamiltonian"

$$H_{\rm M} \equiv \int d^4x\, (i)\frac{\delta}{\delta A_\kappa(x)} g_{\kappa\lambda}\Big[(i\hbar)\frac{\delta}{\delta A_\lambda(x)} + K^\lambda(x; A)\Big] \tag{3.13}$$

and the "drift force" is given by

$$\begin{aligned} K^\lambda(x; A) &\equiv \frac{\delta S_{\rm YM}}{\delta A_\lambda(x)} + a^{-1}D^\lambda \partial \cdot A \\ &= D_\mu F^{\mu\lambda} + a^{-1}D^\lambda \partial^\mu A_\mu. \end{aligned} \tag{3.14}$$

The linear part of this force is

$$\partial_\kappa\, g^{\kappa\mu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + a^{-1}\partial_\nu\, g^{\kappa\mu}\partial_k A_\mu$$

which, for $a > 0$ defines a regular hyperbolic operator that is invertible with Feynman boundary conditions. As above, one may show that the solutions to this equation for different values of the gauge-parameter a are gauge equivalent to each other, and for $a \to \infty$, one regains the formal weight $N \exp(iS_{\text{YM}})/\hbar$.

The drift force $D_\lambda \partial \cdot A$ is not conservative, so one cannot write down an exact solution to the time-independent Fokker-Planck equation $H_{\text{FP}}P = 0$. Nor can one express the solution as a functional integral over a local 4-dimensional action. However we shall, by successive changes of variable, transform this equation into an equation of Dyson-Schwinger type that may be used for perturbative expansion and non-perturbative solution.

4. Quantum effective action in stochastic quantization

The partition function $Z(J)$, which is the generating functional of correlation functions with source J, is defined by

$$Z(J) \equiv \int dA \ \exp(J_x A_x / \hbar) \ P(A). \tag{4.1}$$

It is the fourier transform (with respect to iJ_x) of the probability distribution $P(A)$, and satisfies the fourier-transformed time-independent Fokker-Planck equation,

$$J_x \left[J_x - K_x(\hbar \frac{\partial}{\partial J}) \right] Z(J) = 0. \tag{4.2}$$

Here $K_x(\hbar \frac{\partial}{\partial J})$ is the local cubic polynomial in its argument $\hbar \frac{\partial}{\partial J}$ that is defined in (3.11). We set $Z(J) = \exp[W(J)/\hbar]$, where the "free energy" $W(J)$ is the generating functional of connected correlation functions, in terms of which the time-independent Fokker-Planck equation reads

$$J_x \left[J_x - K_x \left(\frac{\partial W}{\partial J} + \hbar \frac{\partial}{\partial J} \right) 1 \right] = 0. \tag{4.3}$$

The quantum effective action

$$\Gamma(A_{\text{cl}}) = J_x A_{\text{cl},x} - W(J) \tag{4.4}$$

is obtained by Legendre transformation from $W(J)$, by inverting

$$A_{\text{cl},x}(J) \equiv \frac{\partial W}{\partial J_x} = \frac{\hbar}{Z} \frac{\partial Z}{\partial J_x} = \langle A_x \rangle_J, \tag{4.5}$$

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to obtain $J_x = J_x(A_{cl})$. In the following we shall write $\Gamma(A)$ instead of $\Gamma(A_{cl})$ when there is no ambiguity caused by using the same symbol for the quantum Euclidean field A and the classical source $A = A_{cl}$. The gluon propagator in the presence of the source J is given by

$$\mathcal{D}_{xy}(J) \equiv \hbar^{-1}\langle\, (A_x - \langle A_x\rangle_J)\,(A_y - \langle A_y\rangle_J)\,\rangle_J$$
$$= \frac{\partial A_y}{\partial J_x} = \frac{\partial^2 W}{\partial J_x \partial J_y}. \tag{4.6}$$

We note in passing that the gluon propagator $\mathcal{D}_{xy}(J)$ in the presence of the source J is a positive matrix, since one has, for any f_x, $\sum_{xy} f_x \mathcal{D}_{xy}(J) f_y = \hbar^{-1}\langle\, X^2\,\rangle_J \geq 0$, where $X \equiv \sum_x f_x(A_x - \langle A_x\rangle_J)$, which is positive since it is the expectation-value of a square. It is expressed in terms of the Legendre-transformed variables A and $\Gamma(A)$ by

$$\mathcal{D}^{-1}{}_{xy}(A) = \frac{\partial^2\Gamma(A)}{\partial A_x \partial A_y}. \tag{4.7}$$

Expectation-values of functionals $O = O(A)$ are expressed in terms of $Z(J)$, $W(J)$ or $\Gamma(A)$ by

$$\langle O\rangle_J = Z^{-1}\, O\!\left(\hbar\frac{\partial}{\partial J}\right) Z$$
$$= O\!\left(\frac{\partial W}{\partial J} + \hbar\frac{\partial}{\partial J}\right)1 \tag{4.8}$$
$$\langle O\rangle_A = O\!\left(A + \hbar\mathcal{D}(A)\frac{\partial}{\partial A}\right)1,$$

where the subscript indicates that the expectation-value is calculated in the presence of the source J or A. In the last line, the argument of O is written in matrix notation, and reads explicitly $A_x + \hbar\mathcal{D}_{xy}(A)\frac{\partial}{\partial A_y}$.

The gluon propagator $\mathcal{D}_{xy}(J)$ is a positive matrix, as is its inverse $\mathcal{D}^{-1}{}_{xy}(A)$, so both $W(J)$ and $\Gamma(A)$ are convex functionals. Physics is regained when the source J is set to 0, namely $J_x = \frac{\partial\Gamma}{\partial A_x} = 0$. Since $\Gamma(A)$ is a convex functional, the point $\frac{\partial\Gamma}{\partial A_x} = 0$ is an absolute minimum of Γ. In the absence of spontaneous symmetry breaking, this minimum is unique and defines the quantum vacuum. Thus physics is regained at the absolute minimum of $\Gamma(A)$, which justifies the name 'quantum effective action'.

In terms of the Legendre-transformed variables, the time-independent Fokker-Planck equation (4.3) reads

$$\frac{\partial\Gamma}{\partial A_x}\left[\frac{\partial\Gamma}{\partial A_x} + K_x\!\left(A + \hbar\mathcal{D}(A)\frac{\partial}{\partial A}\right)1\right] = 0. \tag{4.9}$$

Here $\mathcal{D}(A)$ is expressed in terms of $\Gamma(A)$ by (4.7), and $K_x\!\left(A + \hbar\mathcal{D}(A)\frac{\partial}{\partial A}\right)1$ is evaluated next.

5. Quantum effective drift force

We call

$$\mathcal{Q}_x(A) \equiv K_x\Big(A + \hbar \mathcal{D}(A)\frac{\partial}{\partial A}\Big)1. \tag{5.1}$$

the 'quantum effective drift force'. It is the expectation-value

$$\mathcal{Q}_x(A_{\text{cl}}) = \langle K_x \rangle_{A_{\text{cl}}} \tag{5.2}$$

of the drift force (3.11) in the presence of the source A_{cl}, as one sees from (4.8). To evaluate it, we expand $K_x(A)$ in terms of its coefficient functions

$$K_x(A) = K^{(1)}_{xy} A_y + (2!)^{-1} K^{(2)}_{x;yz} A_y A_z + (3!)^{-1} K^{(3)}_{xyzw} A_y A_z A_w. \tag{5.3}$$

The coefficient functions are found from (3.11), and are given in the explicit notation by

$$\begin{aligned}
K^{(1)}{}^{ab}_{\kappa\lambda}(x,y) &= - S^{(2)}{}^{ab}_{\text{YM}\kappa\lambda}(x,y) \\
&= \delta^{ab}\,[\,(\partial^2 \delta_{\kappa\lambda} - \partial_\kappa \partial_\lambda) + a^{-1}\partial_\kappa \partial_\lambda\,]\delta(x-y)
\end{aligned} \tag{5.4}$$

$$K^{(2)}{}^{abc}_{\kappa\lambda\mu}(x;y,z) = - S^{(3)}{}^{abc}_{\text{YM}\kappa\lambda\mu}(x,y,z) + a^{-1}K^{(2)}{}^{abc}_{\text{gt}\ \kappa\lambda\mu}(x;y,z) \tag{5.5}$$

$$\begin{aligned}
- S^{(3)}{}^{abc}_{\text{YM}\kappa\lambda\mu}(x,y,z) = gf^{abc}\Big(&\partial_{[\lambda}\delta(x-y)\delta_{\mu]\kappa}\,\delta(y-z) + \partial_{[\mu}\delta(y-z)\delta_{\kappa]\lambda}\,\delta(z-x) \\
&+ \partial_{[\kappa}\delta(z-x)\delta_{\lambda]\mu}\,\delta(x-y)\Big)
\end{aligned} \tag{5.6}$$

$$K^{(2)}{}^{abc}_{\text{gt}\ \kappa\lambda\mu}(x;y,z) = gf^{abc}\,\partial_{[\mu}\delta(x-z)\delta_{\lambda]\kappa}\,\delta(x-y) \tag{5.7}$$

$$\begin{aligned}
K^{(3)}{}^{abcd}_{\kappa\lambda\mu\nu}(x,y,z,w) &= - S^{(4)}{}^{abcd}_{\text{YM}\kappa\lambda\mu\nu}(x,y,z,w) \\
&= -g^2\Big(f^{abe}f^{cde}\delta_{\kappa[\mu}\delta_{\nu]\lambda} + f^{ace}f^{bde}\delta_{\kappa[\lambda}\delta_{\nu]\mu} \\
&\quad + f^{ade}f^{cbe}\delta_{\kappa[\mu}\delta_{\lambda]\nu}\Big)\,\delta(x-y)\delta(x-z)\delta(x-w),
\end{aligned} \tag{5.8}$$

where $\delta_{\kappa[\lambda}\delta_{\nu]\mu} \equiv \delta_{\kappa\lambda}\delta_{\nu\mu} - \delta_{\kappa\nu}\delta_{\lambda\mu}$ etc. The contribution to each coefficient $K^{(n)}$ from S_{YM} is symmetric in all its arguments, including the first. Thus $K^{(3)}{}^{abcd}_{\kappa\lambda\mu\nu}(x,y,z,w)$ is symmetric under permutations of its 4 arguments. Moreover $K^{(1)}{}^{ab}_{\kappa\lambda}(x,y)$ is manifestly symmetric in its arguments. On the other hand the first argument of $K^{(2)}{}^{abc}_{\text{gt}\ \kappa\lambda\mu}(x;y,z)$ is distinguished.

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The evaluation of the quantum effective drift force, $\mathcal{Q}_x(A) = K_x(A + \hbar\mathcal{D}\frac{\partial}{\partial A})1$, is straightforward. By substitution into (5.3) we have

$$
\begin{aligned}
\mathcal{Q}_x(A) &= K_{xy}^{(1)}A_y + (2!)^{-1}K_{x;yz}^{(2)}\left(A_y + \hbar\mathcal{D}_{yu}\frac{\partial}{\partial A_u}\right)A_z \\
&\quad + (3!)^{-1}\ K_{xyzw}^{(3)}\left(A_y + \hbar\mathcal{D}_{yu}\frac{\partial}{\partial A_u}\right)\left(A_z + \hbar\mathcal{D}_{zv}\frac{\partial}{\partial A_v}\right)A_w \\
&= K_{xy}^{(1)}A_y + (2!)^{-1}K_{x;yz}^{(2)}(A_yA_z + \hbar\mathcal{D}_{yz}) \\
&\quad + (3!)^{-1}\ K_{xyzw}^{(3)}\left(A_y + \hbar\mathcal{D}_{yu}\frac{\partial}{\partial A_u}\right)(A_zA_w + \hbar\mathcal{D}_{zw}).
\end{aligned}
\tag{5.9}
$$

Use of the identity,

$$
\frac{\partial\mathcal{D}_{zw}(A)}{\partial A_r} = -\mathcal{D}_{zs}(A)\mathcal{D}_{wt}(A)\ \frac{\partial^3\Gamma(A)}{\partial A_r\partial A_s\partial A_t},
\tag{5.10}
$$

that follows from $(\mathcal{D}^{-1})_{z,w}(A) = \frac{\partial^2\Gamma(A)}{\partial A_z\partial A_w}$, gives the formula for $\mathcal{Q}_x(A)$ that is the first equation of next section.

6. Basic equations for $\mathcal{Q}$ and Γ

The first basic equation of the present method is the formula, just derived, for the quantum effective drift force,

$$
\begin{aligned}
\mathcal{Q}_x(A) &= K_x(A) + \hbar\ (2!)^{-1}K_{x;yz}^{(2)}\mathcal{D}_{yz} + \hbar\ (2!)^{-1}\ K_{xyzw}^{(3)}\mathcal{D}_{yz}A_w \\
&\quad - \hbar^2\ (3!)^{-1}\ K_{xyzw}^{(3)}\mathcal{D}_{yr}\mathcal{D}_{zs}\mathcal{D}_{wt}\ \frac{\partial^3\Gamma(A)}{\partial A_r\partial A_s\partial A_t},
\end{aligned}
\tag{6.1}
$$

where $\mathcal{D} = \mathcal{D}(A)$ is the gluon propagator in the presence of the source A, and is expressed in terms of $\Gamma(A)$ by $(\mathcal{D}^{-1})_{z,w} = \frac{\partial^2\Gamma(A)}{\partial A_z\partial A_w}$. This equation is represented graphically in fig. 1. The terms of order $\hbar$ and $\hbar^2$ correspond to one and two loops in the figure, and we write

$$
\mathcal{Q}_x = K_x + \hbar\mathcal{Q}_{1\text{loop},x}(\Gamma) + \hbar^2\mathcal{Q}_{2\text{loop},x}(\Gamma).
\tag{6.2}
$$

The second basic equation of the present approach is obtained by writing the time-independent Fokker-Planck equation (4.9), satisfied by the quantum effective action, Γ, in terms of the quantum effective drift force, $\mathcal{Q}_x(A)$,

$$
\frac{\partial\Gamma}{\partial A_x}\left[\frac{\partial\Gamma}{\partial A_x} + \mathcal{Q}_x(A)\right] = 0.
\tag{6.3}
$$

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This equation is of classic Hamilton-Jacobi type, with energy $E = 0$, and hamiltonian $H(p, A) = p_x[p_x + \mathcal{Q}_x(A)]$.

The pair of equations (6.1) and (6.3) forms the basis of the present approach and allows a systematic calculation of the correlation functions. Equation (6.1) resembles the DS equation for the gluon field in Faddeev-Popov theory namely $\frac{\partial \Gamma}{\partial A_x} = \frac{\partial S}{\partial A_x}(A + \hbar \mathcal{D} \frac{\partial}{\partial A})1$, where $S = S_{\mathrm{YM}} + S_{\mathrm{gf}} + S_{\mathrm{gh}}$, and S_{gh} is the ghost action. Indeed the same expressions appear in both equations, as is seen most easily from fig. 1, except that the contribution from the ghost action, $\frac{\delta S_{\mathrm{gh}}}{\delta A_\mu}(A + \hbar \mathcal{D} \frac{\delta}{\delta A})1$, is replaced by the term proportional to a^{-1} in the gluon vertex $K^{(2)}$.

In the functional equations (6.1) and (6.3), satisfied by $\mathcal{Q}_x(A)$ and $\Gamma(A)$, A is a dummy variable, and each of these functional equations represents a set of equations satisfied by the coefficient functions that appear in the expansions in powers of A,

$$\mathcal{Q}_x(A) = Q^{(1)}_{x;y} A_y + (2!)^{-1} Q^{(2)}_{x;y_1,y_2} A_{y_1} A_{y_2} + \ldots \tag{6.4}$$

$$\Gamma(A) = (2!)^{-1}\Gamma^{(2)}_{y_1,y_2} A_{y_1} A_{y_2} + (3!)^{-1}\Gamma^{(3)}_{y_1,y_2,y_3} A_{y_1} A_{y_2} A_{y_3} + \ldots \ , \tag{6.5}$$

where $\Gamma^{(n)}$ is the proper n-vertex. The individual equations for the coefficient functions are conveniently obtained by differentiating (6.1) and (6.3) n times with respect to A_z, and then setting $A = 0$.

We now come to an important point. The time-independent Fokker-Planck equation (4.9) satisfied by $\Gamma(A)$ is equivalent to the pair of coupled equations (6.1) and (6.3) that is satisfied by the pair $\Gamma(A)$ and $\mathcal{Q}_x(A)$. Indeed *every solution of (6.1) and (6.3) yields a solution of (4.9) and conversely.* This remark is the key to transforming the time-independent Fokker-Planck equation into an equation of DS type. For it turns out that the Hamilton-Jacobi equation (6.3) may be solved exactly and explicitly for the coefficient functions $\Gamma^{(n)}$ of $\Gamma(A)$ in terms of the coefficient functions $Q^{(m)}_x$ of $\mathcal{Q}_x(A)$, where $m < n$. In fact we shall obtain a simple algebraic – indeed, rational – formula for $\Gamma^{(n)} = \Gamma^{(n)}(Q)$ for every n. This allows us to change variable from the quantum effective action, $\Gamma = \Gamma(\mathcal{Q})$, to the quantum effective drift force, $\mathcal{Q}_x$. It will be the last in our series of changes of variable, $P(A) \to Z(J) \to W(J) \to \Gamma(A) \to \mathcal{Q}_x(A)$.

Neither the Hamilton-Jacobi equation (6.3) nor its solution $\Gamma = \Gamma(\mathcal{Q})$ contains $\hbar$. When the solution of (6.3), $\Gamma = \Gamma(\mathcal{Q})$, is substituted into (6.2), one obtains an equation of the form

$$\mathcal{Q}_x = K_x + \hbar \mathcal{Q}_{1\mathrm{loop},x}(\mathcal{Q}) + \hbar^2 \mathcal{Q}_{2\mathrm{loop},x}(\mathcal{Q}). \tag{6.6}$$

This is an equation of DS type for the quantum effective drift force $\mathcal{Q}_x$. By iteration, it provides the $\hbar$-expansion of $\mathcal{Q}_x$. The zeroth-order term K_x, given in (3.11), is a local function of A. We shall find an approximate, non-perturbative solution of this equation. But first we must find $\Gamma(\mathcal{Q})$.

7. Solution for quantum effective action $\Gamma(\mathcal{Q})$

In this section we solve (6.3) for the coefficient functions $\Gamma^{(2)} = \Gamma^{(2)}(Q)$ and $\Gamma^{(3)} = \Gamma^{(3)}(Q)$. The solution for $\Gamma^{(4)}$ and $\Gamma^{(n)}$ for arbitrary n is found in Appendix D.

The solution for $\Gamma^{(2)}$, reads simply

$$\Gamma^{(2)}_{x_1 x_2} = - Q^{(1)}_{x_1; x_2}. \tag{7.1}$$

Note: By definition, $\Gamma^{(n)}_{x_1 x_2 \ldots x_n}$ is symmetric in its n arguments, whereas $Q^{(n-1)}_{x_1; x_2, \ldots x_n}$ has a distinguished first argument and is symmetric only in the remaining $n-1$ arguments, so in general the equation $\Gamma^{(n)}_{x_1 x_2 \ldots x_n} = - Q^{(n-1)}_{x_1; x_2, \ldots x_n}$ would not be consistent. However symmetries in fact constrain $Q^{(1)}_{x_1; x_2}$ to be symmetric, $Q^{(1)}_{x_1; x_2} = Q^{(1)}_{x_2; x_1}$, as we will see, so (7.1) is in fact consistent.

To prove (7.1), we differentiate (6.3) with respect to A_{x_1} and A_{x_2}, and obtain, after setting $A = 0$,

$$\Gamma^{(2)}_{x_1 x} \, (\Gamma^{(2)}_{x x_2} + Q^{(1)}_{x; x_2}) + (x_1 \leftrightarrow x_2) = 0. \tag{7.2}$$

To solve this equation for $\Gamma^{(2)}$, we diagonalize all the matrices by taking fourier transforms. In the extended notation this equation reads,

$$\int d^4 x \, \Big(\Gamma^{(2)\, a_1 a}_{\mu_1 \mu}(x_1, x)[\Gamma^{(2)\, a a_2}_{\mu \mu_2}(x, x_2) \; + Q^{(1)\, a a_2}_{\mu \mu_2}(x; x_2)]$$
$$+ [(x_1, \mu_1, a_1) \leftrightarrow (x_2, \mu_2, a_2)] \Big) = 0, \tag{7.3}$$

and we take fourier transforms,

$$Q^{(1)\, ab}_{\lambda \mu}(x; y) = \delta^{ab} \, (2\pi)^{-4} \int d^4 k \exp[ik \cdot (x - y)] \, \tilde{Q}^{(1)}_{\lambda \mu}(k)$$
$$\Gamma^{(2)\, ab}_{\lambda \mu}(x, y) = \delta^{ab} \, (2\pi)^{-4} \int d^4 k \exp[ik \cdot (x - y)] \, \tilde{\Gamma}^{(2)}_{\lambda \mu}(k). \tag{7.4}$$

Color, translation, and Lorentz invariance, and use of the transverse and longitudinal projectors, $P_{\lambda\mu}^{\mathrm{T}}(k) = (\delta_{\lambda\mu} - k_\lambda k_\mu/k^2)$ and $P_{\lambda\mu}^{\mathrm{L}}(k) = k_\lambda k_\mu/k^2$ give the decomposition of these quantities into their transverse and longitudinal invariant functions,

$$\tilde{Q}_{\lambda\mu}^{(1)}(k) = Q^{(1)\mathrm{T}}(k^2) \, P_{\lambda\mu}^{\mathrm{T}}(k) + Q^{(1)\mathrm{L}}(k^2) \, P_{\lambda\mu}^{\mathrm{L}}(k)$$
$$\tilde{\Gamma}_{\lambda\mu}^{(2)}(k) = T(k^2) \, P_{\lambda\mu}^{\mathrm{T}}(k) + a^{-1}L(k^2) \, P_{\lambda\mu}^{\mathrm{L}}(k). \tag{7.5}$$

The coefficient a^{-1} is introduced here for later convenience. In terms of the fourier transforms, (7.3) reads

$$\tilde{\Gamma}_{\mu_1\mu}^{(2)}(k) \, [\tilde{\Gamma}_{\mu\mu_2}^{(2)}(k) + \tilde{Q}_{\mu\mu_2}^{(1)}(k)] \; + (k, \mu_1, \mu_2 \leftrightarrow - k, \mu_2, \mu_1) = 0. \tag{7.6}$$

Color and Lorentz symmetries, as expressed in (7.5), constrain $\tilde{Q}_{\lambda\nu}^{(1)}(k)$ to be a symmetric tensor that is even in k, $\tilde{Q}_{\lambda\nu}^{(1)}(k) = \tilde{Q}_{\nu\lambda}^{(1)}(k) = \tilde{Q}_{\lambda\nu}^{(1)}(-k)$, as is $\tilde{\Gamma}_{\lambda\nu}^{(2)}(k)$. Products of such tensors have the same property, and as a result, the two terms in (7.6) are equal, and we have

$$\tilde{\Gamma}_{\mu_1\mu}^{(2)}(k) \, [\tilde{\Gamma}_{\mu\mu_2}^{(2)}(k) + \tilde{Q}_{\mu\mu_2}^{(1)}(k)] \; = 0, \tag{7.7}$$

which proves the assertion (7.1). For future reference, we note

$$(\tilde{D}^{-1})_{\lambda\mu}(k) = \tilde{\Gamma}_{\lambda\mu}^{(2)}(k) = - \, \tilde{Q}_{\lambda\mu}^{(1)}(k)$$
$$= T(k^2) \, P_{\lambda\mu}^{\mathrm{T}}(k) + a^{-1}L(k^2) \, P_{\lambda\mu}^{\mathrm{L}}(k). \tag{7.8}$$

Here we have introduced the usual gluon propagator, with sources set to 0, $D_{xy} = \mathcal{D}_{xy}(A)|_{A=0}$. It is given in terms of Γ by $(D^{-1})_{xy} = \frac{\partial^2\Gamma}{\partial A_x \partial A_y}|_{A=0} = \Gamma_{xy}^{(2)}$.

We next find $\Gamma^{(3)}$. For this purpose we differentiate (6.3) with respect to A_{x_1}, A_{x_2} and A_{x_3}, and obtain, after setting $A = 0$,

$$\Gamma_{x_1x}^{(2)} \, (\Gamma_{xx_2x_3}^{(3)} + Q_{x;x_2x_3}^{(2)}) \; + \text{(cyclic)} = 0, \tag{7.9}$$

where we have used $\Gamma^{(2)} = - \, Q^{(1)}$, and (cyclic) represents the cyclic permutations of (1,2,3). A novelty of the stochastic method is now apparent. For $Q_{x;x_2,x_3}^{(2)}$, unlike $Q_{x;x_2}^{(1)}$, is not completely symmetric in all its arguments as it would be if the drift force were conservative. As a result, the equation $\Gamma_{x,x_2,x_3}^{(3)} + Q_{x;x_2,x_3}^{(2)} = 0$ has no solution. This is already apparent to zero order in $\hbar$, where $Q_{x;x_2,x_3}^{(2)} = K_{x;x_2,x_3}^{(2)}$, but $K_{x;x_2,x_3}^{(2)}$ is not symmetric in its 3 arguments, as noted above.

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To solve (7.9) for $\Gamma^{(3)}_{xx_2x_3}$, we again diagonalize the matrix $\Gamma^{(2)}_{x_1x}$ by Fourier transformation. To do so, we write the last equation in the extended notation,

$$\int d^4x \; \left(\; \Gamma^{(2)a_1a}_{\mu_1\mu}(x_1,x) \; [\Gamma^{(3)aa_2a_3}_{\mu\mu_2\mu_3}(x,x_2,x_3) + Q^{(2)aa_2a_3}_{\mu\mu_2\mu_3}(x;x_2,x_3)] \; + \; (\text{cyclic}) \; \right) = 0, \tag{7.10}$$

and take fourier transforms,

$$Q^{(2)a_1a_2a_3}_{\mu_1\mu_2\mu_3}(x_1;x_2,x_3) = (2\pi)^{-8} \int d^4k_1 d^4k_3 d^4k_3 \exp(ik_1\cdot x_1 + ik_2\cdot x_2 + ik_3\cdot x_3)$$
$$\times \; \delta(k_1 + k_2 + k_3) \; \tilde{Q}^{(2)a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1;k_2,k_3) \tag{7.11}$$

$$\Gamma^{(3)a_1a_2a_3}_{\mu_1\mu_2\mu_3}(x_1,x_2,x_3) = (2\pi)^{-8} \int d^4k_1 d^4k_3 d^4k_3 \exp(ik_1\cdot x_1 + ik_2\cdot x_2 + ik_3\cdot x_3)$$
$$\times \; \delta(k_1 + k_2 + k_3) \; \tilde{\Gamma}^{(3)a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3), \tag{7.12}$$

where $\tilde{Q}^{(2)a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1;k_2,k_3)$ and $\tilde{\Gamma}^{(3)a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)$ are defined only for $k_1+k_2+k_3 = 0$. This gives

$$\tilde{\Gamma}^{(2)}_{\mu_1\mu}(k_1) \; [\tilde{\Gamma}^{(3)a_1a_2a_3}_{\mu\mu_2\mu_3}(k_1,k_2,k_3) + \tilde{Q}^{(2)a_1a_2a_3}_{\mu\mu_2\mu_3}(k_1;k_2,k_3)] \; + (\text{cyclic}) = 0. \tag{7.13}$$

We use the symmetry of $\tilde{\Gamma}^{(3)a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)$ in its three arguments to write this as

$$[\tilde{\Gamma}^{(2)}_{\mu_1\nu_1}(k_1)\delta_{\mu_2\nu_2}\delta_{\mu_3\nu_3} + (\text{cyclic})] \; \tilde{\Gamma}^{(3)a_1a_2a_3}_{\nu_1\nu_2\nu_3}(k_1,k_2,k_3) = - \; H^{(3)a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3), \tag{7.14}$$

where

$$H^{(3)a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) \equiv \tilde{\Gamma}^{(2)}_{\mu_1\mu}(k_1) \; \tilde{Q}^{(2)a_1a_2a_3}_{\mu\mu_2\mu_3}(k_1;k_2,k_3) + (\text{cyclic}). \tag{7.15}$$

To complete the diagonalization of $\tilde{\Gamma}^{(2)}_{\lambda\mu}(k)$, and solve (7.14) for $\tilde{\Gamma}^{(3)a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)$, we apply a transverse or longitudinal projector to each of its three arguments, and use the transverse and longitudinal decomposition of $\tilde{\Gamma}^{(2)}_{\lambda\mu}(k)$ given in (7.8). One obtains $\tilde{\Gamma}^{(3)a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)$ in terms of its transverse and longitudinal projections, defined by $X^T_\mu(k) \equiv P^T_{\mu\nu}(k)X_\nu(k)$ and $X^L_\mu(k) \equiv P^L_{\mu\nu}(k)X_\nu(k)$,

$$\tilde{\Gamma}^{(3)\text{TTT}a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) = - \; [T(k_1^2) + T(k_2^2) + T(k_3^2)]^{-1} \; H^{(3)\text{TTT}a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) \tag{7.16}$$

$$\tilde{\Gamma}^{(3)\text{LTT}a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) = - \; [a^{-1}L(k_1^2) + T(k_2^2) + T(k_3^2)]^{-1} \; H^{(3)\text{LTT}a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) \tag{7.17}$$

etc. The corresponding formulas for $\tilde{\Gamma}^{(4)}$ and $\tilde{\Gamma}^{(n)}$ are found in Appendix D.

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8. Dyson-Schwinger equation for the gluon propagator

We have solved the second basic equation (6.3) for the coefficient functions $\tilde{\Gamma}^{(n)}$, and expressed them in terms of the $\tilde{Q}^{(m)}$, for $m < n$. We now turn to the first basic equation (6.1), and derive the equations for the coefficient functions $Q^{(m)}$ by the same method of taking derivatives and setting $A = 0$. To derive the equation for $Q^{(1)}$, we differentiate (6.1) with respect to A_y, and obtain, after setting $A = 0$,

$$
\begin{aligned}
Q^{(1)}_{x;y} = {} & K^{(1)}_{x;y} - \hbar \, (2!)^{-1} \, K^{(2)}_{x;x_1,x_2} D_{x_1 y_1} \, D_{x_2 y_2} \, \Gamma^{(3)}_{y_1 y_2 y} + \hbar \, (2!)^{-1} \, K^{(3)}_{xx_1 x_2 y} \, D_{x_1 x_2} \\
& - \hbar^2 \, (3!)^{-1} K^{(3)}_{xx_1 x_2 x_3} \, D_{x_1 y_1} \, D_{x_2 y_2} \, D_{x_3 y_3} \, \Gamma^{(4)}_{y_1 y_2 y_3 y} \\
& + \hbar^2 \, (2!)^{-1} \, K^{(3)}_{xx_1 x_2 x_3} D_{x_1 z_1} D_{x_2 z_2} \, \Gamma^{(3)}_{z_1 z_2 z_3} D_{z_3 y_1} \, D_{x_3 y_2} \, \Gamma^{(3)}_{y_1, y_2, y},
\end{aligned}
\tag{8.1}
$$

where we have again used (5.10). This equation is represented diagrammatically in fig. 2.

In momentum space the coefficients (5.4) – (5.8) of the drift force read

$$
K^{(1)\,ab}_{\lambda\mu}(x;y) = \delta^{ab} \, (2\pi)^{-4} \int d^4 k \exp[ik \cdot (x - y)] \, \tilde{K}^{(1)}_{\lambda\mu}(k),
\tag{8.2}
$$

$$
\begin{aligned}
K^{(2)\,a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(x_1; x_2, x_3) = {} & f^{a_1 a_2 a_3} \, (2\pi)^{-8} \int d^4 k_1 d^4 k_3 d^4 k_3 \exp(ik_1 \cdot x_1 + ik_2 \cdot x_2 + ik_3 \cdot x_3) \\
& \times \delta(k_1 + k_2 + k_3) \, \tilde{K}^{(2)}_{\mu_1 \mu_2 \mu_3}(k_1; k_2, k_3)
\end{aligned}
\tag{8.3}
$$

$$
\begin{aligned}
K^{(3)\,a_1 a_2 a_3 a_4}_{\mu_1 \mu_2 \mu_3 \mu_4}(x_1, x_2, x_3, x_4) = {} & (2\pi)^{-12} \int d^4 k_1 d^4 k_3 d^4 k_3 d^4 k_4 \exp(i \sum_{i=1}^{4} k_i \cdot x_i) \\
& \times \delta(k_1 + k_2 + k_3 + k_4) \, \tilde{K}^{(3)\,a_1 a_2 a_3 a_4}_{\mu_1 \mu_2 \mu_3 \mu_4},
\end{aligned}
\tag{8.4}
$$

where

$$
- \tilde{K}^{(1)}_{\lambda\mu}(k) = [\, (k^2 \delta_{\lambda\mu} - k_\lambda k_\mu) + a^{-1} k_\lambda k_\mu \,]
\tag{8.5}
$$

$$
\begin{aligned}
& \tilde{K}^{(2)}_{\mu_1 \mu_2 \mu_3}(k_1; k_2, k_3) = - \tilde{S}^{(3)}_{\text{YM}\mu_1 \mu_2 \mu_3}(k_1, k_2, k_3) + a^{-1} \tilde{K}^{(2)}_{\text{gt}\,\mu_1 \mu_2 \mu_3}(k_1; k_2, k_3), \\
& - \tilde{S}^{(3)}_{\text{YM}\mu_1 \mu_2 \mu_3}(k_1, k_2, k_3) \equiv ig \, [\, (k_1)_{[\mu_2} \delta_{\mu_3]\mu_1} + (\text{cyclic}) \,] \\
& \tilde{K}^{(2)}_{\text{gt}\,\mu_1 \mu_2 \mu_3}(k_1; k_2, k_3) \equiv i \, g \, [\, (k_3)_{\mu_3} \delta_{\mu_1 \mu_2} - (2 \leftrightarrow 3) \,].
\end{aligned}
\tag{8.6}
$$

$$
\begin{aligned}
- \tilde{K}^{(3)a_1 a_2 a_3 a_4}_{\mu_1 \mu_2 \mu_3 \mu_4} = g^2 \, (\, & f^{a_1 a_2 e} f^{a_3 a_4 e} \, \delta_{\mu_1[\mu_3} \delta_{\mu_4]\mu_2} + f^{a_1 a_3 e} f^{a_2 a_4 e} \, \delta_{\mu_1[\mu_2} \delta_{\mu_4]\mu_3} \\
& + f^{a_1 a_4 e} f^{a_3 a_2 e} \, \delta_{\mu_1[\mu_3} \delta_{\mu_2]\mu_4} \,).
\end{aligned}
\tag{8.7}
$$

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With $\tilde{Q}^{(1)} = -\tilde{D}^{-1}$, we obtain finally the DS equation for the gluon propagator

$$
\begin{aligned}
-\delta^{ab}\,(\tilde{D}^{-1})_{\lambda\mu}(k) = {} & -\delta^{ab}\,[(k^2\delta_{\kappa\lambda} - k_\kappa k_\lambda) + a^{-1}k_\kappa k_\lambda] \\
& -\hbar\,f^{aa_1a_2}\,(2!)^{-1}(2\pi)^{-4}\int dk_1\;\tilde{K}^{(2)}_{\lambda\lambda_1\lambda_2}(k; -k_1, k_1 - k) \\
& \qquad \times \tilde{D}_{\lambda_1\mu_1}(k_1)\,\tilde{D}_{\lambda_2\mu_2}(k - k_1)\,\tilde{\Gamma}^{(3)\,a_1a_2b}_{\quad\mu_1\mu_2\mu}(k_1, k - k_1, -k) \\
& +\hbar\,(2!)^{-1}(2\pi)^{-4}\int dk_1\;\tilde{K}^{(3)\,a\;c\;c\;b}_{\quad\lambda\lambda_1\lambda_2\mu}\,\tilde{D}_{\lambda_1\lambda_2}(k_1) \\
& +\delta^{ab}\,\tilde{Q}^{(1)}_{2l,\lambda\mu}(k),
\end{aligned}
\tag{8.8}
$$

where the two-loop term is given by

$$
\begin{aligned}
\delta^{ab}\,\tilde{Q}^{(1)}_{2l,\lambda\mu}(k) \equiv {} & -\hbar^2\,(3!)^{-1}(2\pi)^{-8}\int dk_1 dk_2\;\tilde{K}^{(3)\,aa_1a_2a_3}_{\quad\lambda\lambda_1\lambda_2\lambda_3}\,\tilde{D}_{\lambda_1\mu_1}(k_1)\,\tilde{D}_{\lambda_2\mu_2}(k_2) \\
& \qquad \times \tilde{D}_{\lambda_3\mu_3}(k - k_1 - k_2)\,\tilde{\Gamma}^{(4)\,a_1a_2a_3b}_{\quad\mu_1\mu_2\mu_3\mu}(k_1, k_2, k - k_1 - k_2, -k) \\
& +\hbar^2\,(2!)^{-1}(2\pi)^{-8}\int dk_1 dk_2\;\tilde{K}^{(3)\,aa_1a_2a_3}_{\quad\lambda\lambda_1\lambda_2\lambda_3}\,\tilde{D}_{\lambda_1\nu_1}(k_1) \\
& \qquad \times \tilde{D}_{\lambda_2\nu_2}(k_2)\,\tilde{\Gamma}^{(3)\,a_1a_2b_1}_{\quad\nu_1\nu_2\nu_3}(k_1, k_2, -k_1 - k_2)\,\tilde{D}_{\nu_3\mu_1}(k_1 + k_2) \\
& \qquad \times \tilde{D}_{\lambda_3\mu_2}(k - k_1 - k_2)\,\tilde{\Gamma}^{(3)\,b_1a_3b}_{\quad\mu_1\mu_2\mu}(k_1 + k_2, k - k_1 - k_2, -k).
\end{aligned}
\tag{8.9}
$$

9. Truncation scheme

To obtain a non-perturbative solution of the DS equations, it is necessary to truncate them in some way. Needless to say, truncation remains an uncontrolled approximation until it is tested by varying the scheme, or by comparison with numerical simulation, as discussed in the Introduction and Conclusion. Moreover the truncation scheme is gauge dependent. This situation is familiar in atomic physics where bound state calculations are done in the Coulomb gauge. We shall ultimately solve the truncated system in the Landau-gauge limit.

As a first step we neglect the two-loop contribution in eq. (8.8). We shall also not retain the tadpole term, which in any case gets absorbed in the renormalization. The 3-vertex that we will obtain

$$
\tilde{\Gamma}^{(3)\,a_1a_2a_3}_{\quad\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = f^{a_1a_2a_3}\,\tilde{\Gamma}^{(3)}_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3),
\tag{9.1}
$$

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defined for $k_1 + k_2 + k_3 = 0$, has the color dependence that allows us to use the identity $f^{aa_1a_2}f^{a_1a_2b} = N\delta^{ab}$ for $SU(N)$ color group. As a result, the DS equation (8.8) simplifies to

$$
\begin{aligned}
(\tilde{D}^{-1})_{\lambda\mu}(k) = \ & (k^2\delta_{\kappa\lambda} - k_\kappa k_\lambda) + a^{-1}k_\kappa k_\lambda \\
& + \hbar N \ (2!)^{-1}(2\pi)^{-4}\int dk_1 \ \tilde{K}^{(2)}_{\lambda\lambda_1\lambda_2}(k; -k_1, k_1 - k), \\
& \times \tilde{D}_{\lambda_1\mu_1}(k_1) \ \tilde{D}_{\lambda_2\mu_2}(k - k_1) \ \tilde{\Gamma}^{(3)}_{\mu_1\mu_2\mu}(k_1, k - k_1, -k).
\end{aligned}
\tag{9.2}
$$

We convert this into a DS equation for the invariant propagator functions $T(k^2)$ and $L(k^2)$. The gluon propagator is given by

$$
\tilde{D}_{\lambda\mu}(k) = \frac{P^T_{\lambda\mu}(k)}{T(k^2)} + a \ \frac{P^L_{\lambda\mu}(k)}{L(k^2)}.
\tag{9.3}
$$

To get the DS equation for $T(k^2)$, we apply projectors $P^T_{\kappa,\nu}(k)$ to both free indices of (9.2), and obtain $[P^T(k)\tilde{D}^{-1}(k)P^T(k)]_{\lambda\mu} = T(k^2)P^T_{\lambda\mu}(k)$ on the left hand side. We take the trace on Lorentz indices in d space-time dimensions, and use $P^T_{\lambda\lambda}(k^2) = d - 1$, to obtain the DS equation for $T(k^2)$,

$$
T(k^2) = \ k^2 + \frac{\hbar N}{2(d-1)(2\pi)^d}\int d^d k_1 \ [I^{T,TT}(k_1, k) + 2I^{T,TL}(k_1, k) + I^{T,LL}(k_1, k)], \tag{9.4}
$$

where

$$
I^{T,TT}(k_1, k) = \frac{\tilde{K}^{(2)\,TT\,T}_{\lambda\lambda_1\lambda_2}(k, -k_1, -k_2) \ \tilde{\Gamma}^{(3)\,T\,T\,T}_{\lambda_1\lambda_2\lambda}(k_1, k_2, -k)}{T(k_1^2) \ T(k_2^2)}
\tag{9.5}
$$

$$
I^{T,TL}(k_1, k) = a \ \frac{\tilde{K}^{(2)\,TT\,L}_{\lambda\lambda_1\lambda_2}(k; -k_1, -k_2) \ \tilde{\Gamma}^{(3)\,T\,L\,T}_{\lambda_1\lambda_2\lambda}(k_1, k_2, -k)}{T(k_1^2) \ L(k_2^2)}
\tag{9.6}
$$

$$
I^{T,LL}(k_1, k) = a^2 \ \frac{\tilde{K}^{(2)\,TL\,L}_{\lambda\lambda_1\lambda_2}(k; -k_1, -k_2) \ \tilde{\Gamma}^{(3)\,L\,L\,T}_{\lambda_1\lambda_2\lambda}(k_1, k_2, -k)}{L(k_1^2) \ L(k_2^2)},
\tag{9.7}
$$

$k_2 = k - k_1$, and the transverse and longitudinal projections are defined in sec. 7.

Similarly, to get the DS equation for $L(k^2)$, we apply projectors $P^L_{\kappa,\nu}(k)$ to both free indices of (9.2), and obtain $[P^L(k)\tilde{D}^{-1}(k)P^L(k)]_{\lambda\mu} = a^{-1}L(k^2)P^L_{\lambda\mu}(k)$ on the left hand side. We take the trace on Lorentz indices in d space-time dimensions, and use $P^L_{\lambda\lambda}(k^2) = 1$, to obtain the DS equation for $L(k^2)$,

$$
a^{-1}L(k^2) = \ a^{-1}k^2 + \frac{\hbar N}{2(2\pi)^d}\int d^d k_1 \ [I^{L,TT}(k_1, k) + 2I^{L,TL}(k_1, k) + I^{L,LL}(k_1, k)], \tag{9.8}
$$

where

$$I^{L,TT}(k_1, k) = \frac{\tilde{K}^{(2)\,LT\,T}_{\lambda\lambda_1\lambda_2}(k, -k_1, -k_2)\ \tilde{\Gamma}^{(3)\,T\,T\,L}_{\lambda_1\lambda_2\lambda}(k_1, k_2, -k)}{T(k_1^2)\ T(k_2^2)} \tag{9.9}$$

$$I^{L,TL}(k_1, k) = a\ \frac{\tilde{K}^{(2)\,LT\,L}_{\lambda\lambda_1\lambda_2}(k; -k_1, -k_2)\ \tilde{\Gamma}^{(3)\,T\,L\,L}_{\lambda_1\lambda_2\lambda}(k_1, k_2, -k)}{T(k_1^2)\ L(k_2^2)} \tag{9.10}$$

$$I^{L,LL}(k_1, k) = a^2\ \frac{\tilde{K}^{(2)\,LL\,L}_{\lambda\lambda_1\lambda_2}(k; -k_1, -k_2)\ \tilde{\Gamma}^{(3)\,L\,L\,L}_{\lambda_1\lambda_2\lambda}(k_1, k_2, -k)}{L(k_1^2)\ L(k_2^2)}. \tag{9.11}$$

The vertex $\tilde{K}^{(2)}$ is given in (8.6). To complete the truncation scheme and obtain closed equations for $T(k^2)$ and $L(k^2)$, we need an approximation for the vertex $\tilde{\Gamma}^{(3)}$. We will approximate $\tilde{\Gamma}^{(3)}$ by its value to zero-order in $\hbar$. This vertex is expressed linearly in terms of $\tilde{Q}^{(2)}$ by the exact formulas of sec. 7, which may be written $\tilde{\Gamma}^{(3)} = M\tilde{Q}^{(2)}$, where $M = M(\tilde{D})$. At tree level, $\tilde{Q}^{(2)}$ is given by

$$\begin{aligned}\tilde{Q}^{(2)} &= \tilde{K}^{(2)} \\ &= -\tilde{S}^{(3)}_{\mathrm{YM}} + a^{-1}\tilde{K}^{(2)}_{\mathrm{gt}}\end{aligned} \tag{9.12}$$

where we have used (6.2) and (8.6). Each of these terms contributes additively to $\tilde{\Gamma}^{(3)} = M\tilde{Q}^{(2)}$. Moreover $\tilde{S}^{(3)}_{\mathrm{YM}}$, is symmetric in all its arguments. As a result, it contributes unchanged to $\tilde{\Gamma}^{(3)}$, as one sees from (7.9), and we have

$$\tilde{\Gamma}^{(3)}_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = \tilde{S}^{(3)}_{\mathrm{YM}\,\mu_1\mu_2\mu_3}(k_1, k_2, k_3) + \tilde{\Gamma}^{(3)}_{\mathrm{gt}\,\mu_1\mu_2\mu_3}(k_1, k_2, k_3), \tag{9.13}$$

where $\tilde{\Gamma}^{(3)}_{\mathrm{gt}} = M\tilde{K}^{(2)}_{\mathrm{gt}}$ is obtained from (7.15) – (7.17) by the substitutions

$$\begin{aligned}\tilde{Q}^{(2)}_{\mu_1\mu_2\mu_3}(k_1; k_2, k_3) &\to a^{-1}\tilde{K}^{(2)}_{\mathrm{gt}\,\mu_1\mu_2\mu_3}(k_1; k_2, k_3) \\ \tilde{\Gamma}^{(3)}_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3) &\to \tilde{\Gamma}^{(3)}_{\mathrm{gt}\,\mu_1\mu_2\mu_3}(k_1, k_2, k_3).\end{aligned} \tag{9.14}$$

Finally, to obtain $\tilde{\Gamma}^{(3)}_{\mathrm{gt}}$ to zero-order in $\hbar$, we substitute the tree-level propagators

$$T(k^2) \to k^2, \qquad L(k^2) \to k^2, \tag{9.15}$$

into the formulas of sec. 7. This is done in Appendix E, and gives for the vertex $\tilde{\Gamma}^{(3)}_{\mathrm{gt}}$,

$$\tilde{\Gamma}^{(3)\,T\,T\,T}_{\mathrm{gt}\,\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = 0$$

$$\tilde{\Gamma}^{(3)\,T\,T\,L}_{\mathrm{gt}\,\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = -ig\ \frac{k_1^2 - k_2^2}{ak_1^2 + ak_2^2 + k_3^2}\ (k_3)_{\mu_3}\ [P^{\mathrm{T}}(k_1)P^{\mathrm{T}}(k_2)]_{\mu_1\mu_2}$$

$$\tilde{\Gamma}^{(3)\,T\,L\,L}_{\mathrm{gt}\,\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = -ia^{-1}g\ \left(\ \frac{k_3^2 - ak_1^2}{ak_1^2 + k_2^2 + k_3^2}\ (k_2)_{\mu_2}\ [P^{\mathrm{T}}(k_1)P^{\mathrm{L}}(k_3)]_{\mu_1\mu_3} - (2 \leftrightarrow 3)\ \right)$$

$$\tilde{\Gamma}^{(3)\,L\,L\,L}_{\mathrm{gt}\,\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = -ia^{-1}g\ \left(\ \frac{k_2^2 - k_3^2}{k_1^2 + k_2^2 + k_3^2}\ (k_1)_{\mu_1}\ [P^{\mathrm{L}}(k_2)P^{\mathrm{L}}(k_3)]_{\mu_2\mu_3} + (\text{cyclic})\right),$$

$$\tag{9.16}$$

valid to zero-order in $\hbar$. Because of the denominators, the vertex $\tilde{\Gamma}^{(3)}_{\text{gt}}$ is non-local even to this order. Equations (9.13) and (9.16) complete the specification of $\tilde{\Gamma}^{(3)}$ that appears in the truncated DS equations (9.4) and (9.8) for the 2 invariant propagator functions $T(k^2)$ and $L(k^2)$.

In Faddeev-Popov theory there are, by contrast, 3 invariant propagator functions, namely, these 2 plus the ghost propagator. However in Faddeev-Popov theory, the Slavnov-Taylor identity in its BRST version implies that the gluon self-energy is transverse, so there are finally only 2 independent invariant propagator functions in Faddeev-Popov theory also, namely, the transverse part of the inverse gluon propagator and the ghost propagator.[9] In the present theory, the longitudinal part of the gluon propagator replaces the ghost propagator as the second invariant propagator function. There is no BRST symmetry in the present theory, but it possesses a Ward identity, derived in Appendix C, that expresses the effect of a gauge transformation and constrains the form of divergences.

10. Landau gauge limit

We now specialize to the Landau gauge limit $a \to 0$. We cannot directly set $a = 0$ in the DS equations (9.4) and (9.8) because both vertices contain terms of order a^{-1}. With the gluon propagator given by (9.3), we take as an Ansatz that the invariant propagator functions $T(k^2)$ and $L(k^2)$ remain finite in the limit $a \to 0$. This accords with the behavior obtained in [6] by a Born-Oppenheimer type calculation. At $a = 0$, the propagator is indeed transverse, which is the defining condition for the Landau gauge, and $L(k^2)$ does drop out of the propagator. However the vertices contain terms of order a^{-1}, and, remarkably, the longitudinal propagator function $L(k^2)$ does not decouple at $a = 0$, but remains an essential component of the dynamics!

We next determine the a-dependence of the vertices asymptotically, at small a. By (8.6), we have $K^{(2)} = -S^{(3)}_{\text{YM}} + a^{-1}K^{(2)}_{\text{gt}}$, so this vertex contains a term of order a^0 and a term of order a^{-1}. We take the asymptotic limit of (9.16) at small a, and obtain the interesting a-dependence

$$\begin{aligned}
\tilde{\Gamma}^{(3)\,\text{T T T}}_{\text{gt}\;\mu_1\mu_2\mu_3}(k_1, k_2, k_3) &= 0 \\
\tilde{\Gamma}^{(3)\,\text{T T L}}_{\text{gt}\;\mu_1\mu_2\mu_3}(k_1, k_2, k_3) &= \tilde{\gamma}^{\text{T T L}}_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3) \\
\tilde{\Gamma}^{(3)\,\text{T L L}}_{\text{gt}\;\mu_1\mu_2\mu_3}(k_1, k_2, k_3) &= a^{-1}\, \tilde{\gamma}^{\text{T L L}}_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3) \\
\tilde{\Gamma}^{(3)\,\text{L L L}}_{\text{gt}\;\mu_1\mu_2\mu_3}(k_1, k_2, k_3) &= a^{-1}\, \tilde{\gamma}^{\text{L L L}}_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3),
\end{aligned} \tag{10.1}$$

---

[9] In practice the truncated DS equations in Faddeev-Popov theory violate the Slavnov-Taylor identities to some extent.

where

$$\tilde{\gamma}^{\mathrm{T\,T\,L}}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) = -\,ig\,\frac{k_1^2-k_2^2}{k_3^2}\,(k_3)_{\mu_3}\,[P^{\mathrm{T}}(k_1)P^{\mathrm{T}}(k_2)]_{\mu_1\mu_2}$$

$$\tilde{\gamma}^{\mathrm{T\,L\,L}}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) = -\,ig\,\frac{k_3^2}{k_2^2+k_3^2}\,(k_2)_{\mu_2}\,[P^{\mathrm{T}}(k_1)P^{\mathrm{L}}(k_3)]_{\mu_1\mu_3}\,-\,(2\leftrightarrow 3) \qquad (10.2)$$

$$\tilde{\gamma}^{\mathrm{L\,L\,L}}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) = -\,ig\,\frac{k_2^2-k_3^2}{k_1^2+k_2^2+k_3^2}\,(k_1)_{\mu_1}\,[P^{\mathrm{L}}(k_2)P^{\mathrm{L}}(k_3)]_{\mu_2\mu_3}\,+\,(\mathrm{cyclic})$$

are independent of a. These quantities are anti-symmetric in their three arguments so, for example, $\tilde{\gamma}^{\mathrm{T\,T\,L}}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) = -\tilde{\gamma}^{\mathrm{T\,L\,T}}_{\mu_1\mu_3\mu_2}(k_1,k_3,k_2)$, etc. We see that $\tilde{\Gamma}^{(3)}$ also contains a term of order a^0 and a term of order a^{-1}.

The DS equation for $L(k^2)$, eq. (9.8), is consistent with our Ansatz in the Landau gauge limit only if the leading term on the right is also of order a^{-1}. This is non-trivial, because both vertices contain terms of order a^{-1}, so in principle terms of order a^{-2} could appear on the right hand side which would invalidate our Ansatz.

We now derive the DS equation for $L(k^2)$ in the Landau-gauge limit by evaluating in succession the terms (i) $I^{\mathrm{L,TT}}$, (ii) $I^{\mathrm{L,LL}}$, and (iii) $I^{\mathrm{L,TL}}$ that appear on the right hand side of (9.8), in the limit $a \to 0$.

(i) Consider eq. (9.9) for $I^{\mathrm{L,TT}}$. It contains no explicit powers of a. Moreover the vertex $\tilde{\Gamma}^{(3)\,\mathrm{T\,T\,L}}_{\mu_1\mu_2\mu}(k_1,k_2,-k)$, given in (10.1), is of order a^0. Thus the inconsistency of a term of order a^{-2} is avoided, and this intermediate state will give a contribution of required order a^{-1} only if the vertex, $\tilde{K}^{(3)\,\mathrm{LT\,T}}_{\lambda\lambda_1\lambda_2}(k;-k_1,-k_2)$, gives a contribution of order a^{-1}. The term $a^{-1}\tilde{K}^{(3)}_{\mathrm{gt}\,\lambda\lambda_1\lambda_2}(k;-k_1,-k_2)$ in (8.6) is in fact of this order. The projected components of $\tilde{K}^{(2)}_{\mathrm{gt}}$ are easily read off (8.6) by writing $\delta_{\lambda\mu} = [P^{T}(k)+P^{L}(k)]_{\lambda\mu}$, which gives

$$\tilde{K}^{(2)}_{\mathrm{gt}\,\mu_1\mu_2\mu_3}(k_1;k_2,k_3) = ig\,(k_3)_{\mu_3}\,\Big(\,[\,P^{T}(k_1)+P^{L}(k_1)\,][\,P^{T}(k_2)+P^{L}(k_2)\,]\,\Big)_{\mu_1,\mu_2}$$
$$-\,(2\leftrightarrow 3). \qquad (10.3)$$

The polarization vector $(k_3)_{\mu_3}$ is purely longitudinal, as is $(k_2)_{\mu_2}$, and this implies

$$\tilde{K}^{(2)\,\mathrm{L\,T\,T}}_{\mathrm{gt}\,\lambda\lambda_1\lambda_2}(k;-k_1,-k_2) = \tilde{K}^{(2)\,\mathrm{T\,T\,T}}_{\mathrm{gt}\,\lambda\lambda_1\lambda_2}(k;-k_1,-k_2) = 0. \qquad (10.4)$$

Thus there is no contribution of the required order a^{-1} from $I^{\mathrm{L,TT}}$.

(ii) Consider eq. (9.11) for $I^{\mathrm{L,LL}}$. It has the coefficient a^2, so there is no contribution of the required order a^{-1} from $I^{\mathrm{L,LL}}$ either.

(iii) Now consider eq. (9.10) for $I^{\mathrm{L,TL}}$. It has the coefficient a. So when each vertex is of order a^{-1} there is an overall contribution to $a^{-1}L(k^2)$ of the required order a^{-1}. As a result, the DS equation (9.8) for $L(k^2)$ simplifies in Landau gauge to

$$L(k^2) = k^2 + \hbar N \ (2\pi)^{-4} \int d^4k_1 \ \frac{\tilde{K}_{\mathrm{gt}\ \lambda\lambda_1\lambda_2}^{(2)\mathrm{L\ T\ L}}(k; -k_1, -k_2) \ \tilde{\gamma}_{\lambda_1\lambda_2\lambda}^{\mathrm{T\ L\ L}}(k_1, k_2, -k)}{T(k_1^2) \ L(k_2^2)}. \tag{10.5}$$

By (10.3) and (10.2), we have

$$\tilde{K}_{\mathrm{gt}\ \lambda\lambda_1\lambda_2}^{(2)\mathrm{L\ T\ L}}(k; -k_1, -k_2) \ \tilde{\gamma}_{\lambda_1\lambda_2\lambda}^{\mathrm{T\ L\ L}}(k_1, k_2, -k) = -\frac{g^2 \ k_2^2}{k_2^2 + k^2} \ [k \cdot P^T(k_1) \cdot k + k \cdot P^T(k_1) \cdot k_2]$$

$$= -2 \ \frac{g^2 \ k_2^2}{k_2^2 + k^2} \ k \cdot P^T(k_1) \cdot k, \tag{10.6}$$

where we have used $k_2 = k - k_1$. Note that a factor of the external momentum k appears at each vertex. This corresponds to the factorization of external ghost momentum in the Landau gauge in Faddeev-Popov theory. This gives the truncated DS equation for $L(k^2)$ in Landau gauge,

$$L(k^2) = k^2 - \frac{2\hbar g^2 N}{(2\pi)^4} \int d^4k_1 \ \frac{k_2^2 \ [k^2 k_1^2 - (k \cdot k_1)^2]}{k_1^2 \ (k_2^2 + k^2) \ T(k_1^2) \ L(k_2^2)}. \tag{10.7}$$

The DS equation for $T(k^2)$, eq. (9.4), is consistent with our Ansatz in the Landau gauge limit only if the leading term on the right is also of order a^0. This is non-trivial, because both vertices contain terms of order a^{-1}, so in principle terms of order a^{-1} and a^{-2} could appear on the right hand side which would invalidate our Ansatz.

We now derive the DS equation for $T(k^2)$ in the Landau-gauge limit by evaluating in succession the terms (i) $I^{\mathrm{T,TT}}$, (ii) $I^{\mathrm{T,TL}}$, and (iii) $I^{\mathrm{T,LL}}$ that appear on the right hand side of (9.4), in the limit $a \to 0$.

(i) Consider eq. (9.5) for $I^{\mathrm{T,TT}}$. It contains no explicit powers of a. By (10.1) and (10.4), the vertices from $\tilde{K}_{\mathrm{gt}\ \lambda\lambda_1\lambda_2}^{(2)\mathrm{TT\ T}}(k; -k_1, -k_2)$ and $\tilde{\Gamma}_{\mathrm{gt}\ \mu_1\mu_2\mu}^{(3)\mathrm{T\ T\ T}}(k_1, k_2, -k)$ vanish, and we obtain from (8.6) and (9.13),

$$\tilde{K}_{\lambda\lambda_1\lambda_2}^{(2)T\ T\ T}(k; -k_1, -k_2) = -\tilde{S}_{\mathrm{YM}\lambda\lambda_1\lambda_2}^{(3)\ TT\ T}(k, -k_1, -k_2) \tag{10.8}$$

and

$$\tilde{\Gamma}_{\mu_1\mu_2\mu}^{(3)T\ T\ T}(k_1, k_2, -k) = \tilde{S}_{\mathrm{YM}\mu_1\mu_2\mu}^{(3)\ T\ T\ T}(k_1, k_2, -k). \tag{10.9}$$

This gives a contribution of the required order.

(ii) Next consider eq. (9.6) for $I^{\mathrm{T,TL}}$. It has coefficient a. By (9.13) and (10.1), we have

$$\tilde{\Gamma}^{(3)\,\mathrm{T\,L\,T}}_{\mu_1\mu_2\mu}(k_1, k_2, -k) = \tilde{S}^{(3)\,\mathrm{T\,L\,T}}_{\mu_1\mu_2\mu}(k_1, k_2, -k) + \tilde{\gamma}^{(3)\,\mathrm{T\,L\,T}}_{\mu_1\mu_2\mu}(k_1, k_2, -k), \tag{10.10}$$

which is of order a^0. Thus only that part of the vertex $\tilde{K}^{(2)\,\mathrm{TT\,L}}_{\lambda\lambda_1\lambda_2}(k; -k_1, -k_2)$ that is of order a^{-1} will contribute to the desired order a^0. However from (10.2) for $\tilde{\gamma}^{(3)\,\mathrm{T\,T\,L}}_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3)$, and by evaluation of $\tilde{S}^{(3)\,\mathrm{T\,L\,T}}_{\mu_1\mu_2\mu}(k_1, k_2, -k)$, one obtains

$$\tilde{\Gamma}^{(3)\,\mathrm{T\,L\,T}}_{\mu_1\mu_2\mu}(k_1, k_2, -k) = 0. \tag{10.11}$$

(iii) Finally consider eq. (9.7) for $I^{\mathrm{T,LL}}$. It has coefficient a^2. To get a net contribution of order a^0, we make the substitutions of the relevant projected vertices,

$$\begin{aligned}
\tilde{K}^{(2)\,\mathrm{TL\,L}}_{\lambda\lambda_1\lambda_2}(k; -k_1, -k_2) &\to a^{-1}\tilde{K}^{(2)\,\mathrm{TL\,L}}_{\mathrm{gt}\,\lambda\lambda_1\lambda_2}(k; -k_1, -k_2) \\
\tilde{\Gamma}^{(3)\,\mathrm{L\,L\,T}}_{\mu_1\mu_2\mu}(k_1, k_2, -k) &\to a^{-1}\tilde{\gamma}^{(3)\,\mathrm{L\,L\,T}}_{\mu_1\mu_2\mu}(k_1, k_2, -k),
\end{aligned} \tag{10.12}$$

by (8.6), (9.13), and (10.1). Again the conclusion is consistent with our Ansatz.

We have now found all the terms on the right hand side of (9.4) that contribute to $T(k^2)$ in the Landau gauge limit, namely,

$$I^{T,TT}(k_1, k) = \frac{-\tilde{S}^{(3)\,TT\,T}_{\mathrm{YM}\lambda\lambda_1\lambda_2}(k, -k_1, -k_2)\,\tilde{S}^{(3)\,T\,T\,T}_{\mathrm{YM}\lambda_1\lambda_2\lambda}(k_1, k_2, -k)}{T(k_1^2)\,T(k_2^2)} \tag{10.13}$$

$$I^{T,TL}(k_1, k) = 0 \tag{10.14}$$

$$I^{T,LL}(k_1, k) = \frac{\tilde{K}^{(2)\,\mathrm{TL\,L}}_{\mathrm{gt}\,\lambda\lambda_1\lambda_2}(k; -k_1, -k_2)\,\tilde{\gamma}^{(3)\,\mathrm{L\,L\,T}}_{\lambda_1\lambda_2\lambda}(k_1, k_2, -k)}{L(k_1^2)\,L(k_2^2)}. \tag{10.15}$$

and $k_2 = k - k_1$. The last term is given explicitly by

$$I^{T,LL}(k_1, k) = -g^2\,\frac{(k_1^2 + k_2^2)\,k_1 \cdot P^T(k) \cdot k_2 - k_2^2\,k_1 \cdot P^T(k) \cdot k_1 - k_1^2\,k_2 \cdot P^T(k) \cdot k_2}{(k_1^2 + k_2^2)\,L(k_1^2)\,L(k_2^2)} \tag{10.16}$$

$$I^{T,LL}(k_1, k) = 2\,g^2\,\frac{k_1^2\,k^2 - (k_1 \cdot k)^2}{k^2\,L(k_1^2)\,L(k_2^2)}, \tag{10.17}$$

where we have used $k_1 \cdot P^T(k) \cdot k_1 = k_2 \cdot P^T(k) \cdot k_2 = -k_1 \cdot P^T(k) \cdot k_2$. The non-local denominator $(k_1^2 + k_2^2)^{-1}$ has cancelled out of this expression.

We have obtained a consistent Landau gauge limit of the truncated DS equations for the invariant propagator functions $T(k^2)$ and $L(k^2)$. As asserted, the invariant longitudinal propagator function $L(k^2)$ does not decouple in this limit. The reader will have noticed a striking similarity to the corresponding equations in Faddeev-Popov theory, with the longitudinal propagator replacing the ghost propagator.

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11. Infrared critical exponents

We shall solve the the DS equations (9.4) and (10.7) for the asymptotic forms of $T(k^2)$ and $L(k^2)$ in Landau gauge at low momentum. We suppose that at asymptotically small k, they obey simple power laws,

$$T(k^2) \sim C_T \ (k^2)^{1+\alpha_T}$$
$$L(k^2) \sim C_L \ (k^2)^{1+\alpha_L}, \tag{11.1}$$

where α_T and α_L are infrared critical exponents whose value we wish to determine. Canonical dimension corresponds to $\alpha_T = \alpha_L = 0$. As explained in sec. 3, we know that in the Landau gauge limit, $a \to 0$, the gauge field A is constrained to be transverse $\partial \cdot A = 0$, and to lie inside the Gribov horizon, that is to say, where the Faddeev-Popov operator is positive, $-\partial \cdot D(A) > 0$. The transversality condition is satisfied by our Ansatz. As has been shown many times [22], [26], the positivity condition strongly suppresses the low-momentum components of $\tilde{A}(k)$. Recalling that the transverse part of the gluon propgator is given by $D^T(k^2) = \langle |\tilde{A}(k)|^2 \rangle$, we look for a solution for which $D^T(k^2) = 1/T(k^2)$ is suppressed at low k, so $T(k^2)$ is enhanced at small k compared to the canonical power $T(k^2) = k^2$. This means $\alpha_T < 0$.

We now estimate the power of k of the various terms in the DS equation (9.4) for $T(k^2)$. The analysis is similar to the Faddeev-Popov case [2], [3], [4], [5], [6]. The left hand side has the power $(k^2)^{1+\alpha_T}$. The tree-level term is k^2, so with $\alpha_T < 0$, the tree level term is subdominant in the infrared and may be neglected. To evaluate the loop integral $\int d^4k_1$, asymptotically at low external momentum k we take k to be small compared to a QCD mass scale, $|k| << \Lambda_{\text{QCD}}$, and we rescale the variable of integration according to $k_1^\mu = |k|x^\mu$. We now have a dimensionless integral in which the QCD mass scale appears only in the very small ratio $|k|/\Lambda_{\text{QCD}}$. In the asymptotic infrared limit, this ratio goes to 0, and everywhere in the integrand we use the asymptotic forms (11.1). This is equivalent to using the asymptotic forms (11.1) everywhere in the original integral. We shall see that the resulting integral is convergent, which means that the integral is effectively cut off at momentum $k_1 \sim k$.

We now estimate the contributions of the terms $I^{T,TT}$ and $I^{T,LL}$, eqs. (10.13) and (10.15), to the right hand side of the DS equation (9.4), by simply counting powers of k and k_1. One finds that, after integration $\int d^4k_1$, these terms are of order $(k^2)^{1-2\alpha_T}$ and $(k^2)^{1-2\alpha_L}$ respectively, while the left-hand side is of order $(k^2)^{1+\alpha_T}$, with $\alpha_T < 0$. The

powers match on both sides only if $\alpha_L > 0$. In this case, $I^{T,LL}$ is the dominant term on the right, and by equating powers of k, one obtains

$$\alpha_T = -2\alpha_L, \tag{11.2}$$

and $\alpha_L > 0$. We retain only the dominant term $I^{T,LL}$ on the right in (9.4), which simplifies, for arbitrary space-time dimension d, to

$$C_T \ (k^2)^{1+\alpha_T} = \frac{\hbar g^2 N}{(d-1) \ C_L^2 \ (2\pi)^d \ k^2} \int d^d k_1 \ \frac{k_1^2 \ k^2 - (k_1 \cdot k)^2}{(k_1^2)^{1+\alpha_L} \ (k_2^2)^{1+\alpha_L}}, \tag{11.3}$$

where $k_2 = k - k_1$, and we have used (10.17). This agrees with eq. (6.14) of [6] in Faddeev-Popov theory. We write it as

$$\frac{C_T C_L^2}{\hbar g^2 N} = I_T, \tag{11.4}$$

where I_T is evaluated in Appendix F. We have generalized to arbitrary space-time dimenion d, and we take d in the range $2 < d \leq 4$. By equating powers of k for arbitrary d, we find that the critical exponents are related by

$$\alpha_T + 2\alpha_L = -(4-d)/2. \tag{11.5}$$

The last integral is ultraviolet convergent provided that $\alpha_L > (d-2)/4$, which corresponds to $\alpha_T < -1$. For $d = 4$, we obtain $\alpha_L > 1/2$ as the condition for convergence of the integral.

Now consider the DS equation (10.7) for $L(k^2)$ in the infrared asymptotic limit,

$$C_L(k^2)^{1+\alpha_L} = k^2 \ - \ \frac{2\hbar g^2 N}{C_T C_L \ (2\pi)^4} \int d^4 k_1 \ \frac{k^2 k_1^2 - (k \cdot k_1)^2}{(k_1^2)^{2+\alpha_T} \ (k_2^2)^{\alpha_L} \ (k_2^2 + k^2)}, \tag{11.6}$$

for $d = 4$. By power counting, the integral on the right has the power $(k^2)^{1-\alpha_T-\alpha_L}$. This agrees with the power on the left, provided $\alpha_T = 2\alpha_L$, which is identical to the previous equation. However we have also previously found $\alpha_L > 0$. In this case, the tree level term k^2 is dominant in the infrared, and the equation appears inconsistent. However the degree of divergence of the integral is $2\alpha_L$, so the integral diverges for $\alpha_L > 0$, and a subtraction is required. The integral contains an explicit factor of k^2, and the divergence is of the form Bk^2, where B is an infinite constant. We subtract the integrand at $k = 0$, which makes the integral vanish more rapidly than k^2, and add bk^2 on the right, where b is an arbitrary finite constant. The dominant terms are now the tree level term k^2 and bk^2.

For the equation to be consistent, the subtraction term must precisely cancel the tree-level term, so $b = -1$. This gives

$$C_L(k^2)^{1+\alpha_L} = \frac{2\hbar g^2 N}{C_T C_L \ (2\pi)^4} \int d^4 k_1 \ \frac{k^2 k_1^2 - (k \cdot k_1)^2}{(k_1^2)^{2+\alpha_T}} \left(\frac{1}{(k_1^2)^{\alpha_L} \ k_1^2} - \frac{1}{(k_2^2)^{\alpha_L} \ (k_2^2 + k^2)} \right). \tag{11.7}$$

This integral is also convergent in the infrared for $\alpha_T = -2\alpha_L < 0$. The right hand side now vanishes more rapidly than k^2. This conclusion, agrees with the "horizon condition" [25], and with the confinement criterion of Kugo and Ojima in the BRST framework [27], [28]. Conversely we could have imposed the horizon condition on the DS equation for $L(k^2)$, and derived the suppression of the transverse propagator $1/T(k^2)$ at low momentum.

The subtracted expression on the right is most simply evaluated by continuing in space-time dimension d. In this case one can ignore the subtraction term, and evaluate the unsubtracted integral with dimensional regularization for $d < 4$, and continue the resulting expression to $d = 4$,

$$C_L(k^2)^{1+\alpha_L} = - \frac{2\hbar g^2 N}{C_T C_L \ (2\pi)^d} \int d^d k_1 \ \frac{k^2 k_1^2 - (k \cdot k_1)^2}{(k_1^2)^{2+\alpha_T} (k_2^2)^{\alpha_L} (k_2^2 + k^2)}. \tag{11.8}$$

The denominator $k_2^2 + k^2$ results from the non-local expression for the vertex. One obtains the corresponding equation for the ghost propagator in Faddeev-Popov theory, eq. (6.15) of [6], from this equation by the substitution $\frac{2}{k_2^2 + k^2} \to \frac{1}{k_2^2}$.

By equating powers of k for general space-time dimension d, one again gets (11.5), and we see that the DS equations for the transverse and longitudinal parts are consistent. The degree of divergence of this integral is $2\alpha_L$, and after one subtraction its degree of divergence is $2\alpha_L - 2$, so the subtracted integral is convergent provided that $\alpha_L < 1$, or equivalently that $\alpha_T > -2 - (4 - d)/2$. From this and our previous bound, we conclude that for $d = 4$, this subtracted integral and (11.3) are both finite provided that α_L is in the range $1/2 < \alpha_L < 1$, or equivalently that α_T is in the range $-2 < \alpha_T < -1$. We write the preceding equation as

$$\frac{C_T C_L^2}{\hbar g^2 N} = I_L, \tag{11.9}$$

where I_L is evaluated in Appendix F.

Upon comparison with (11.4), we obtain

$$I_T(\alpha_L) = I_L(\alpha_L) \tag{11.10}$$

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which determines the critical exponent α_L. From Appendix F, this gives, for $d = 4$,

$$\frac{\Gamma^2(2 - \alpha_L)\,\Gamma(2\alpha_L - 1)}{\Gamma^2(1 + \alpha_L)\,\Gamma(4 - 2\alpha_L)} = \frac{3\,(-\alpha_L^2 + 2\alpha_L + 2)\,\Gamma^2(1 - \alpha_L)\,\Gamma(2\alpha_L + 1)}{\alpha_L\,\Gamma(\alpha_L + 2)\Gamma(\alpha_L + 3)\Gamma(2 - 2\alpha_L)}. \tag{11.11}$$

Both expressions are finite and positive, as they should be, for α_L in the interval $1/2 < \alpha_L < 1$. Moreover at $\alpha_L = 1/2$, the left-hand side diverges whereas the right is finite. On the other hand at $\alpha_L = 1$, the left-hand side is finite whereas the right diverges. Consequently there is at least one root in the interval $1/2 < \alpha_L < 1$. After cancelling Γ-functions, the last two equations give the quartic equation

$$49\alpha_L^4 - 189\alpha_L^3 + 133\alpha_L^2 + 117\alpha_L - 74 = 0. \tag{11.12}$$

From a numerical investigation it appears that there is only one root in the interval $1/2 < \alpha_L < 1$, with the value

$$\alpha_L \approx 0.5214602698$$
$$\alpha_T = -2\alpha_L \approx -1.04292054. \tag{11.13}$$

12. Conclusion

We derived time-independent stochastic quantization from the principle of gauge equivalence which states that probability distributions that give the same expectation values for all gauge-invariant observables are physically indistiguishable. This quantization is expressed by an equation for the Euclidean proabability distribution $P(A)$ that is of time-independent Fokker-Planck form, with a corresponding equation for the Minkowski case. By making several changes of variable, we transformed this equation into an equation of DS type, suitable for non-perturbative calculations. The most novel of these changes of variable is accomplished when the equation for the quantum effective action Γ is exchanged for an equation for the quantum effective drift force $\mathcal{Q}_x$. We then adopted a truncation scheme and obtained a consistent Landau gauge limit, $a \to 0$, and found, remarkably, that the longitudinal propagator function $L(k^2)$ that appears in the longitudinal part of the gluon propagator $D^L = a/L(k^2)$, does not decouple in the $a \to 0$ limit, but plays a role similar to the ghost in Faddeev-Popov theory.

We calculated the infrared critical exponents that characterize the asymptotic form at low momentum of the transverse and longitudinal components of the gluon propagator in Landau gauge, $D^T \sim 1/(k^2)^{1+\alpha_T}$, and $D^L \sim a/(k^2)^{1+\alpha_L}$, and obtained the values

$\alpha_L \approx 0.5214602698$ and $\alpha_T = -2\alpha_L \approx -1.04292054$. In the Landau-gauge limit $a \to 0$ only the transverse part survives. As a function of k, it vanishes at $k = 0$, albeit rather weakly, $D^T \sim (k^2)^{-1-\alpha_T} \sim (k^2)^{0.043}$. On the other hand, the longitudinal part of the propagator, is long range, $D^L \sim a/(k^2)^{1.521}$. Qualitatively similar values have been obtained recently for the infrared critical exponents of the gluon and ghost propagators in Landau gauge from the DS equation in Faddeev-Popov theory, using a variety of approximations for the vertex, [2], [3], [4], [5] and [6], in particular, [5] and [6], $\alpha_T = -2a_G = -1.1906$, and $a_G = 0.595353$ respectively. As we have argued recently [6], these calculations in Faddeev-Popov theory should be interpreted as including a cut-off at the Gribov horizon. This makes them similar in spirit to the present calculation for which, as shown in sec. 3, the probability also gets concentrated inside the Gribov horizon in the Landau gauge limit $a \to 0$. Reassuringly, the solutions of the DS equation in Faddeev-Popov theory and in the present time-independent stochastic method are in satisfactory agreement.

We comment briefly on the physical significance of our results. (i) We have avoided gauge fixing and instead derived the equation of time-independent stochastic quantization from the principle of *gauge equivalence*, thereby overcoming the Gribov critique. Since we do not gauge *fix*, we do not brutally eliminate "unphysical" variables and keep only "physical" degrees of freedom, which would violate Singer's theorem [29]. Instead, we gently *tame* the gauge degrees of freedom by exploiting the principle of gauge equivalence. (ii) We derived a set of equations of DS type that was solved approximately but non-perturbatively in Landau gauge asymptotically at low momentum. (iii) The values we obtained for the infra-red critical exponents of the gluon propagator in Landau gauge are in satisfactory agreement with corresponding values in Faddeev-Popov theory, and also with numerical simulations. (iv) A striking result of this investigation is that the invariant longitudinal propagator function $L(k^2)$ does not decouple in stochastic quantization, even though the longitudinal part of the gluon propagator $D^L = a/L(k^2)$ vanishes with the gauge parameter a in the Landau gauge limit $a \to 0$. Indeed, because some vertices are of order a^{-1}, transverse gluons exchange longitudinal gluons as virtual particles, with an amplitude that remains finite in the limit $a \to 0$. Thus, while ghosts are absent in time-independent stochastic quantization, they are replaced dynamically by the longitudinal part of the gluon propagator in the Landau gauge limit. In fact, the DS equations (9.4) and (10.7) for $T(k^2)$ and $L(k^2)$ bear a remarkable similarity to the DS equations for the gluon and ghost propagators D and G in Landau gauge in Faddeev-Popov theory, with the

correspondences $D \leftrightarrow 1/T$ and $G \leftrightarrow 1/L$. In both cases, it is the ghost loop or longitudinal-gluon loop that gives the dominant contribution to the transverse-gluon inverse propagator in the infrared region, and causes suppression of the would-be physical, transverse gluon propagator at $k = 0$, a signal that the gluon has left the physical spectrum.

To conclude, we mention some challenging open problems. (i) The possibility of comparison with numerical simulations is an essential and promising aspect of the present situation. Any DS calculation involves a truncation which remains an uncontrolled approximation, without further investigation. It may be controlled by varying the vertex function [5], or by extending the calculation to include the vertex self-consistently. Fortunately, comparison with numerical simulation provides an independent control. In this regard we note that the stochastic quantization used here may be and, in fact, has been effected on the lattice in numerical simulations by Nakamura and collaborators [51],[52], [53], [54], [55]. A direct comparison with this data would require a solution of the DS equations for finite gauge parameter a, or extrapolation of lattice data to $a = 0$. Naturally, a comparison of numerical results with asymptotic infrared calculations also requires control of finite-volume lattice artifacts. (ii) Conversely the results of the DS calculations suggest new numerical calculations. In particular our prediction that, for small values of the gauge parameter a, the longitudinal part of the gluon propagator is long range, should be tested numerically. (iii) The present scheme is not based on a local action, but rather on the DS equations of time-independent stochastic quantization. Renormalizability follows from the indirect argument that correlators of the 4-dimensional time-independent formulation used here coincide with the equal-time correlators of a local, 5-dimensional theory whose renormalizability has been established [49]. The renormalization constants of the 5-dimensional theory were calculated some time ago at the one-loop level, and were found to yield the usual β-function [57], [58], [46]. However a direct proof of renormalizability in the present time-independent formulation remains a challenge. The Ward identity, derived in Appendix C, is a first step. (iv) One should extend the solution obtained here for the asymptotic infrared region to finite momentum k. (v) The Landau gauge is a singular limit, $a \to 0$, of the DS equations for finite gauge parameter a. It would be valuable to also solve the DS equations for arbitrary, finite a. (vi) One should extend the solution of the DS equations to include quarks. (vii) As we have explained, our results are intuitively transparent and lend themselves to a simple confinement scenario in which the would-be-physical transverse gluon leaves the physical spectrum. However it is clear that our discussion of confinement remains at the level of a scenario because we have dealt

here only with the gluon propagator which is a gauge-dependent quantity. This is only a first step in a program, some of whose elements have just been indicated. Clearly the goal is to calculate gauge-invariant quantities. Gauge-invariant states, the hadrons, appear as intermediate states in gluon-gluon and quark-anti-quark scattering amplitudes. One must extend to this sector the solution of the DS equations obtained here, and of the Bethe-Salpeter equations that follow from them.

Appendix A. Time-independent Fokker-Planck equation for quarks

We extend to quarks the derivation of time-independent stochastic quantization from the principle of gauge equivalence. Following the method of sec. 3, we seek a weight function $P = P(A, \psi, \bar{\psi})$ that depends on the gluon and quark and anti-quark fields. We wish to establish a class of gauge-equivalent normalized distributions that includes the formal gauge-invariant weight $P = N \exp(-S)$ as a limiting case. Here

$$S \equiv S_{\mathrm{YM}} + \int d^4x \ \bar{\psi}(m + \not{D})\psi, \tag{A.1}$$

is the Euclidean action of gluons and quarks, $\not{D} \equiv \gamma_\mu D_\mu = \gamma_\mu(\partial_\mu + gA_\mu^a t^a)$, where $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, and the t^a are the quark representation of the Lie algebra of the structure group, $[t^a, t^b] = f^{abc}t^c$. We take P to be the solution of $H_{\mathrm{FP}}P = 0$, where we now specify the extended Fokker-Planck hamiltonian.

As in sec. 3, we take H_{FP} to be of the form

$$\begin{aligned} H_{\mathrm{FP}} &= H_{\mathrm{inv}} - (v, G)^\dagger \\ &= H_{\mathrm{inv}} + \int d^4x \ \left(\ \frac{\delta}{\delta A_\mu^a} D_\mu^{ac} - g\frac{\delta}{\delta\psi}t^c\psi + g\frac{\delta}{\delta\bar{\psi}}(\bar{\psi}t^c) \ \right) v^c, \end{aligned} \tag{A.2}$$

where H_{inv} is a gauge-invariant operator, specified below, that has $\exp(-S)$ as null vector, $H_{\mathrm{inv}} \exp(-S) = 0$, and the Grassmannian deriveratives are left derivatives. Here

$$G^a(x) = - D_\mu^{ac}\frac{\delta}{\delta A_\mu^c(x)} - g(t^a\psi)\frac{\delta}{\delta\psi(x)} + g(\bar{\psi}t^a)\frac{\delta}{\delta\bar{\psi}(x)} \tag{A.3}$$

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is the generator of local gauge transformations that satisfies

$$[G^a(x), A_\mu^b(y)] = -D_\mu^{ab} \, \delta(x - y)$$
$$[G^a(x), \psi(y)] = - \, g \, t^a \psi(x) \, \delta(x - y)$$
$$[G^a(x), \bar{\psi}(y)] = g \, \bar{\psi}(x) t^a \, \delta(x - y) \qquad \text{(A.4)}$$
$$[G^a(x), G^b(y)] = \delta(x - y) \, g f^{abc} G^c(x)$$
$$[G^a(x), H_{\text{inv}}] = 0.$$

With gauge-invariant observables defined by the condition $G^a(x)W = 0$, the proof of sec. 3, that the expectation-value of gauge invariant observables $\langle W \rangle = \int dA d\psi d\bar{\psi} \, W \, P$ is independent of v, applies here as well. As explained in sec. 3, we take $v^a(x) = a^{-1} \partial_\lambda A_\lambda(x)$, where a is a gauge parameter.

There remains to specify H_{inv}. We suppose that it is a sum of gluon and quark and anti-quark hamiltonians,

$$H_{\text{inv}} = H_1 + H_2 + H_3 \qquad \text{(A.5)}$$

where H_1 is the gauge-invariant gluon hamiltonian as in sec. 3,

$$H_1 = \int d^4x \frac{\delta}{\delta A} \Big(-\frac{\delta}{\delta A} - \frac{\delta S}{\delta A} \Big). \qquad \text{(A.6)}$$

For the quark and anti-quark hamiltonians we take

$$H_2 = \int d^4x \frac{\delta}{\delta \psi} \, N_2 \, \Big(\frac{\delta}{\delta \bar{\psi}} + \frac{\delta S}{\delta \bar{\psi}} \Big)$$
$$H_3 = \int d^4x \frac{\delta}{\delta \bar{\psi}} \, N_3 \, \Big(\frac{\delta}{\delta \psi} + \frac{\delta S}{\delta \psi} \Big), \qquad \text{(A.7)}$$

where N_2 and N_3 are gauge-covariant kernels with engineering dimensions of mass. All terms in H_{FP} contain a derivative on the left, which assures that H_{FP} has a null eigenvalue, for we have $\int dA d\psi d\bar{\psi} \, H_{\text{FP}} F = 0$ for any F. The corresponding right eigenvector P that satisfies $H_{\text{FP}} P = 0$, is the physical distribution that we seek, that depends on the gauge parameters. Each of the operators H_i satisfies $H_i \exp(-S) = 0$, so also $H_{\text{inv}} \exp(-S) = 0$. This assures the applicability of the proof of sec. 3, namely that *the normalized solutions for different v are gauge equivalent, $P_v(A) \sim P_{v'}(A)$, and include $N \exp(-S)$ as a limiting distribution.*

We have obtained this result without any assumptions about the kernels N_2 and N_3, apart from gauge covariance (and regularity). This would not be consistent unless *the*

normalized solutions for different choices of the kernels give gauge-equivalent distributions $P_N(A) \sim P_{N'}(A)$ or, in other words, the parameters that specify N_2 are and N_3 are gauge parameters. We prove this directly.

Let δN_2 be an infinitesimal variation of N_2. It induces a corresponding change in H_2

$$\delta H_2 = \int d^4x \frac{\delta}{\delta \psi} \, \delta N_2 \left(\frac{\delta}{\delta \bar{\psi}} + \frac{\delta S}{\delta \bar{\psi}} \right). \tag{A.8}$$

The corresponding change in P satisfies $\delta H_2 P + H_{\mathrm{FP}} \delta P = 0$, so $\delta P = -H_{\mathrm{FP}}^{-1} \delta H_2 P$. Let W be a gauge-invariant observable. We have

$$\begin{aligned}
\delta \langle W \rangle &= \int dA d\psi d\bar{\psi} \, \delta P \, W \\
&= - \int dA d\psi d\bar{\psi} \, H_{\mathrm{FP}}^{-1} \delta H_2 P \, W \\
&= - \int dA d\psi d\bar{\psi} \, P \, \delta H_2^\dagger \, (H_{\mathrm{FP}}^\dagger)^{-1} \, W \\
&= - \int dA d\psi d\bar{\psi} \, P \, \delta H_2^\dagger \, (H_{\mathrm{inv}}^\dagger)^{-1} \, W \\
&= - \int dA d\psi d\bar{\psi} \, \delta H_2 P \, (H_{\mathrm{inv}}^\dagger)^{-1} \, W,
\end{aligned} \tag{A.9}$$

where we have used $(H_{\mathrm{FP}}^\dagger)^{-1} W = (H_{\mathrm{inv}}^\dagger)^{-1} W$ which holds for a gauge-invariant observable, as was shown in sec. 3. Moreover $\delta H_2^\dagger \, (H_{\mathrm{inv}}^\dagger)^{-1} W$ is gauge invariant, so the last expression is independent of the gauge parameter a, as was also shown in sec. 3, and we may evaluate it for $a \to \infty$. We have

$$\lim_{a \to \infty} \delta H_2 P = \int d^4x \frac{\delta}{\delta \psi} \, \delta N_2 \left(\frac{\delta}{\delta \bar{\psi}} + \frac{\delta S}{\delta \bar{\psi}} \right) \lim_{a \to \infty} P = 0, \tag{A.10}$$

because $\lim_{a \to \infty} P \sim N \exp(-S)$. Thus $\delta \langle W \rangle$ vanishes, as asserted.

The quark action satisfies

$$S_{\mathrm{qu}} = \int d^4x \, \bar{\psi}(m + \slashed{D})\psi = - \int d^4x \, \psi C^{-1}(m + \slashed{D})C\bar{\psi}, \tag{A.11}$$

where C is a numerical matrix that acts on spinor and group indices and satisfies $C^{-1}\gamma_\mu C = -\gamma_\mu^{\mathrm{tr}}$ and $C^{-1}t^a C = -(t^a)^{\mathrm{tr}}$, so we have

$$\begin{aligned}
\frac{\delta S}{\delta \bar{\psi}(x)} &= (m + \slashed{D})\psi(x) \\
\frac{\delta S}{\delta \psi(x)} &= -C^{-1}(m + \slashed{D})C\bar{\psi}(x),
\end{aligned} \tag{A.12}$$

where the Grassmannian derivatives are left derivatives. The most general expressions for N_2 and N_3 that are local, gauge-covariant, and have dimension of mass are

$$N_2 = m_2 - b_2 \ \not{D}$$
$$N_3 = - \ C^{-1}(m_3 - b_3 \ \not{D})C, \tag{A.13}$$

where the m_i and b_i are gauge parameters. We expect that the kernel ultimately appears in the denominator in loop integrals so, to improve convergence, we should take $b_2 \neq 0$ and $b_3 \neq 0$. For the gauge choice to respect charge-conjugation invariance, we take $b_2 = b_3$, and $m_1 = m_2 = c \ m$, which gives

$$H_2 = \int d^4x \frac{\delta}{\delta\bar\psi} \ (c \ m - b \ \not{D}) \ \left(\frac{\delta}{\delta\bar\psi} + (m + \not{D})\psi \right)$$
$$H_3 = \int d^4x \frac{\delta}{\delta\bar\psi} \ (-1)C^{-1}(c \ m - b \ \not{D})C \ \left(\frac{\delta}{\delta\psi} + (-1)C^{-1}(m + \not{D})C\bar\psi \right), \tag{A.14}$$

where b and c are gauge parameters. This gauge choice also respects chiral symmetry in the limit $m \to 0$. One may show that the eigenvalues of H_2 and H_3 are the eigenvalues of fermi oscillators, with frequencies λ_n that are the eigenvalues of the operator $(c \ m - b \ \not{D})(\ m + \not{D})$, which for $b = c > 0$, simplifies to $b(m^2 - \ \not{D}^2)$. In this case H_2 and H_3 have the unique null eigenvector $\exp(-S)$, and all other their eigenvalues are strictly positive, as occurs for H_1. Indeed H_1 satisfies

$$\exp(S/2)H_1 \exp(-S/2) = \int d^4x \ \left(\frac{\delta}{\delta A_x} - (1/2)\frac{\delta S}{\delta A_x} \right)\left(- \frac{\delta}{\delta A_x} - (1/2)\frac{\delta S}{\delta A_x} \right), \tag{A.15}$$

where the operator on the right is manifestly positive, with the unique null-vector $\exp(-S/2)$. Thus H_1 has the unique null-vector $\exp(-S)$, and all its other eigenvalues are strictly positive. However we expect that b and c must be kept as independent constants when needed as renormalization counter-terms.

Altogether, the total Fokker-Planck hamiltonian, including quarks, is given by

$$\begin{aligned} H_{\rm FP} = \int d^4x \Big[\ & \frac{\delta}{\delta A_\mu}\Big(- \frac{\delta}{\delta A_\mu} - \frac{\delta S}{\delta A_\mu}\Big) \\ & + \frac{\delta}{\delta\bar\psi} \ (c \ m - b \ \not{D})\Big(\frac{\delta}{\delta\bar\psi}\psi + \frac{\delta S}{\delta\bar\psi} \Big) \\ & + \frac{\delta}{\delta\bar\psi} \ (-1)C^{-1}(c \ m - b \ \not{D})C\Big(\frac{\delta}{\delta\psi} + \frac{\delta S}{\delta\psi} \Big) \\ & + a^{-1} \ \Big(\frac{\delta}{\delta A_\mu^a}D_\mu^{ac} - \ g \ \frac{\delta}{\delta\psi}(t^c\psi) + g \ \frac{\delta}{\delta\bar\psi}(\bar\psi t^c) \Big) \ \partial \cdot A^c \ \Big], \end{aligned} \tag{A.16}$$

where $a > 0$, $b > 0$ and $c > 0$ are gauge parameters.

Appendix B. Time-independent stochastic quantization on the lattice

We briefly outline how to extend time-independent stochastic quantization to lattice gauge theory. To each link (x, μ) of the lattice is associated a variable $U_{x,\mu} \in \mathrm{SU(N)}$. These variables are subject to the local gauge transformation $U_{x,\mu} \to {}^g U_{x,\mu} = g_x^{-1} U_{x,\mu} g_{x+\hat{\mu}}$, where $g_x \in \mathrm{SU(N)}$ is associated to the site x of the lattice. Observables $W(U)$ are invariant under this transformation, $W({}^g U_{x,\mu}) = W(U_{x,\mu})$. Expectation values are calculated by $\langle W \rangle = \int dU\, W(U)\, P_W(U)$, where dU is the product of Haar measure over all link variables of the lattice, and $P_W = N \exp(-S_W)$ is the normalized probability distribution associated to the gauge-invariant Wilson action S_W.

We shall exhibit a Fokker-Planck hamiltonian H_{FP} for the lattice, such that the positive normalized solutions P to $H_{\mathrm{FP}} P = 0$ are gauge equivalent to P_W, $P \sim P_W$. Let $J_{x,\mu}^a$ be the Lie differential operator associated to the group variable on the link (x, μ), that satisfies the Lie algebra commutation relations $[J_{x,\mu}^a, J_{y,\nu}^b] = \delta_{xy}\, \delta_{\mu\nu} f^{abc} J_{x,\mu}^c$. And let G_x be the generator of local gauge transformations that is defined by $(1 + \sum_x \epsilon_x G_x) F(U) = F({}^g U)$, where $g_x = 1 + \epsilon_x$ is an infinitesimal local gauge transformation. These generators satisfie the Lie algebra commutation relations of the local gauge group of the lattice $[G_x^a, G_y^b] = \delta_{xy}\, f^{abc} G_x^c$, and may be expressed as a linear combination of the J's. A hamiltonian with the desired properties is given by

$$
\begin{aligned}
H_{\mathrm{FP}} &= H_{\mathrm{inv}} - (v, G)^{\dagger} \\
H_{\mathrm{inv}} &= \sum_x J_{x,\mu}(\, -J_{x,\mu} - [J_{x,\mu}, S_W]\,) \\
(v, G) &= \sum_x v_x G_x,
\end{aligned}
\tag{B.1}
$$

where $\dagger$ is the adjoint with respect to the inner product dU, and v_x^a is a site variable with values in the Lie algebra. Indeed, the argument of sec. 3 holds here, with the substitution $S_{\mathrm{YM}}(A) \to S_W(U)$, and shows that the probability distributions P_v for different v, defined by $H_{\mathrm{FP}} P = 0$, are gauge equivalent to each other $P_v \sim P_{v'}$ and to P_W. As in sec. 3, we choose $v_x^a(U)$ so the infinitesimal gauge transformation $g_x = 1 + \epsilon\, t^a v_x^a$ is the direction of steepest descent in gauge orbit directions of a minimizing functional $F(U)$. A convenient choice is $F(U) = \sum_{\langle xy \rangle} \mathrm{tr}(I - U_{\langle xy \rangle})$, where the sum extends over all links $\langle xy \rangle$ of the lattice.

Appendix C. Ward Identity

In sec. 3 we showed that probability distributions $P_v(A)$ for different v are gauge equivalent, $P_v(A) \sim P_{v'}(A)$. Another way to make gauge equivalent distributions is by making a local gauge transformation, because, for all gauge-invariant observables $W(A)$, this cannot change the expectation value $\int dA \ W(A)P_v(A) = \int dA \ W(A)P_v({}^g A)$, and we have $P_v(A) \sim P_v({}^g A)$. If the class of gauge-equivalent distributions $P_v(A)$ that was introduced in sec. 3 is large enough, then the gauge transformation corresponds to a change of v,

$$P_v({}^g A) = P_{v'}(A) \tag{C.1}$$

for some v'. This is in fact the case, and provides a Ward identity.

To prove this, we apply the infinitesimal local gauge transformation $1 + (\epsilon, G)$, where $(\epsilon, G) \equiv \int d^4 x \ \epsilon^a(x) G^a(x)$, to the time-independent Fokker-Planck equation (3.5)and (3.6),

$$[1 + (\epsilon, G)] \ [H_{\text{inv}} + (G, v)] \ P_v = 0. \tag{C.2}$$

From the commutation relations

$$\begin{aligned}
[(\epsilon, G), H_{\text{inv}}] &= 0 \\
[(\epsilon, G), G^a(x)] &= -f^{abc} \ \epsilon^b(x) \ G^c(x),
\end{aligned} \tag{C.3}$$

we obtain

$$[(\epsilon, G), (G, v)] = (G, \delta v), \tag{C.4}$$

where

$$\delta v^a \equiv [(\epsilon, G), v^a] + f^{abc} \epsilon^b v^c, \tag{C.5}$$

and, to first order in ϵ,

$$[H_{\text{inv}} + (G, v + \delta v)] \ [1 + (\epsilon, G)] \ P_v = 0. \tag{C.6}$$

Note that while $v^a(x)$ and $\epsilon^a(x)$ are both local gauge transformations, $v^a(x) = v^a(x, A)$ depends on A, but $\epsilon^a(x)$, by assumption, does not. By comparison with the defining equation for the probability distribution $P_{v+\delta v}$,

$$[H_{\text{inv}} + (G, v + \delta v)] \ P_{v+\delta v} = 0, \tag{C.7}$$

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we conclude that the gauge-transformed probability distribution $[1 + (\epsilon, G)] \, P_v(A) = P_v(A + D\epsilon)$ coincides with $P_{v+\delta v}$,

$$P_v(A + D\epsilon) = P_{v+\delta v}(A), \tag{C.8}$$

where δv is given above. This states how a gauge transformation is absorbed by a change in v, and provides the Ward identity.

This identity is inherited by the functionals we introduced, the quantum effective action Γ_v and the quantum effective drift force Q_v, and we have

$$\Gamma_v(A + D\epsilon) = \Gamma_{v+\delta v}(A)$$
$$\mathcal{Q}_v^a(x, A + D\epsilon) = \mathcal{Q}_{v+\delta v}^a(x, A) - f^{abc}\epsilon^b(x)\mathcal{Q}_{v+\delta v}^c(x, A). \tag{C.9}$$

We now specialize to $v = a^{-1}\partial \cdot A$, and obtain

$$\delta v^a = a^{-1}\partial \cdot D^{ac}(A) \, \epsilon^c + a^{-1}f^{abc}\epsilon^b \, \partial \cdot A^c$$
$$= a^{-1}D^{ac}(A) \cdot \partial \, \epsilon^c = a^{-1}[\partial^2 \epsilon^a + f^{abc}A_\mu^b \, \partial_\mu \epsilon^c]. \tag{C.10}$$

Only the derivative of ϵ appears here because, for $v = a^{-1}\partial \cdot A$, the probability distribution $P_v(A)$ is invariant under global (x-independent) gauge transformations. We further specialize to a linear dependence of ϵ on x, $\epsilon^a(x) = \eta_\mu^a x_\mu$, where the η_μ^a are infinitesimal constants. In this case we have

$$\delta v^a = a^{-1}f^{abc}A_\mu^b \, \eta_\mu^c. \tag{C.11}$$

Although this breaks Lorentz invariance, it does not break translational invariance, so the perturbed hamiltonian defined by

$$H_{\mathrm{FP}} + \delta H_{\mathrm{FP}} = H_{\mathrm{inv}} + (G, v) + (G, \delta v)$$
$$= H_{\mathrm{inv}} + a^{-1}(G, \partial \cdot A) + a^{-1}(G, A_\mu \times \eta_\mu), \tag{C.12}$$

where $(A_\mu \times \eta_\mu)^a \equiv f^{abc}A_\mu^b\eta_\mu^c$, is translationally invariant, even though $A_\mu^a + D_\mu^{ac}\epsilon^c = A_\mu^a + \eta_\mu^a + f^{abc}A_\mu^b \, \eta_\nu^c x_\nu$, has an explicit x-dependence. Moreover the inhomogeneous term $a^{-1}\partial^2\epsilon$ in δv vanishes with this choice of ϵ, so $A = 0$ remains the classical vacuum. Without further calculation we conclude that the transformed quantum effective action $\Gamma_v(A + D\epsilon) = \Gamma_{v+\delta v}(A)$ is a translationally invariant functional of A for $v = a^{-1}\partial \cdot A$ and $\epsilon^a(x) = \eta_\mu^a x_\mu$, with $A = 0$ as classical vacuum.

More generally, we note that the gauge field A_μ^a appears undifferentiated in $\delta v^a = a^{-1}f^{abc}A_\mu^b \times \eta_\mu^c$, whereas it is differentiated in $v^a = a^{-1}\partial \cdot A^a$. This means that the perturbation δH_{FP} is softer than the unperturbed hamiltonian H_{FP} or, in other words, less divergent in the ultraviolet. If we calculate with the original hamiltonian, we get a certain number of divergent constants in the correlators. The result of a gauge transformation $\epsilon^a = \eta_\mu^a x_\mu$ on these correlators must agree with a calculation using the soft perturbation. This constrains the divergent constants.

Appendix D. Solution for $\Gamma^{(4)}$ and $\Gamma^{(n)}$

The solution for $\Gamma^{(4)}$ and higher coefficient functions is similar to the solution for $\Gamma^{(3)}$ found in sec. 7. We differentiate (6.3) with respect to A_{x_i} four times and obtain, after setting $A = 0$,

$$
\begin{aligned}
&\Gamma^{(2)}_{x_1,x} \ (\Gamma^{(4)}_{x,x_2,x_3,x_4} + Q^{(3)}_{x;x_2,x_3,x_4}) + \sum \mathrm{part}(4,1) \\
&+ \ \Gamma^{(3)}_{x_1,x_2,x} \ (\Gamma^{(3)}_{x,x_3,x_4} + Q^{(2)}_{x;x_3,x_4}) + \sum \mathrm{part}(4,2) = 0,
\end{aligned}
\tag{D.1}
$$

where $\Gamma^{(2)}$ and $\Gamma^{(3)}$ are already known, and we have again used $\Gamma^{(2)} = -Q^{(1)}$. Here $\sum \mathrm{part}(n,n_1)$ is the sum over all partitions of the set of n objects, $x_1, x_2, ...x_n$, into subsets of n_1 and $n_2 = n - n_1$ objects. In terms of the fourier transforms

$$
\begin{aligned}
Q^{(3)a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}(x_1;x_2,x_3,x_4) = (2\pi)^{-12} \int d^4k_1 d^4k_3 d^4k_3 d^4k_4 \exp(i\sum_{i=1}^{4} k_i \cdot x_i) \\
\times \, \delta(k_1 + k_2 + k_3 + k_4) \, \tilde{Q}^{(3)a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}(k_1;k_2,k_3,k_4)
\end{aligned}
\tag{D.2}
$$

$$
\begin{aligned}
\Gamma^{(4)a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}(x_1,x_2,x_3,x_4) = (2\pi)^{-12} \int d^4k_1 d^4k_3 d^4k_3 d^4k_4 \exp(i\sum_{i=1}^{4} k_i \cdot x_i) \\
\times \, \delta(k_1 + k_2 + k_3 + k_4) \, \tilde{\Gamma}^{(4)a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4),
\end{aligned}
\tag{D.3}
$$

where $\tilde{Q}^{(3)a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}(k_1;k_2,k_3,k_4)$ and $\tilde{\Gamma}^{(4)a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4)$ are defined only for $k_1 + k_2 + k_3 + k_4 = 0$, the equation for $\tilde{\Gamma}^{(4)}$ reads

$$
\tilde{\Gamma}^{(2)}_{\mu_1\nu_1}(k_1) \, \tilde{\Gamma}^{(4)a_1a_2a_3a_4}_{\nu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4) + \sum \mathrm{part}(4,1) = - \, H^{(4)a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_3}(k_1,k_2,k_3,k_4),
\tag{D.4}
$$

where

$$
\begin{aligned}
H^{(4)a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4) \equiv \tilde{\Gamma}^{(2)}_{\mu_1\mu}(k_1) \, \tilde{Q}^{(3)a_1a_2a_3a_4}_{\mu\mu_2\mu_3\mu_4}(k_1;k_2,k_3,k_4) + \sum \mathrm{part}(4,1) \\
+ R^{(4)a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4)
\end{aligned}
\tag{D.5}
$$

$$
\begin{aligned}
R^{(4)a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4) \equiv \tilde{\Gamma}^{(3)a_1a_2a}_{\mu_1\mu_2\mu}(k_1,k_2,-k_3-k_4) \\
\times [\, \tilde{\Gamma}^{(3)aa_3a_4}_{\mu\mu_3\mu_4}(-k_1-k_2,k_3,k_4) + \tilde{Q}^{(2)aa_3a_4}_{\mu\mu_3\mu_4}(-k_1-k_2;k_3,k_4) \,] \\
+ \sum \mathrm{part}(4,2).
\end{aligned}
\tag{D.6}
$$

To solve (D.4), we project on each argument with a transverse transverse or longitudinal projector to obtain

$$\tilde{\Gamma}^{(4)\mathrm{TTTT}a_1a_2a_3a_4}_{\quad\quad\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) = -\,[T(k_1^2) + T(k_2^2) + T(k_3^2) + T(k_4^2)]^{-1}$$
$$\times H^{(4)\mathrm{TTTT}a_1a_2a_3a_4}_{\quad\quad\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4),$$

(D.7)

$$\tilde{\Gamma}^{(4)\mathrm{LTTT}a_1a_2a_3a_4}_{\quad\quad\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) = -\,[a^{-1}L(k_1^2) + T(k_2^2) + T(k_3^2) + T(k_4^2)]^{-1}$$
$$\times H^{(4)\mathrm{LTTT}a_1a_2a_3a_4}_{\quad\quad\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4),$$

(D.8)

etc.

The formula for $\Gamma^{(n)}$ for arbitrary n is similar. Each $\Gamma^{(n)}$ is expressed explicity and uniquely in terms of $Q^{(n-1)}$ and of $\Gamma^{(2)}$ to $\Gamma^{(n-1)}$ which are already known. It is given by a symmetrized sum of products of two factors, as in eqs. (D.5) and (D.6), to which is applied a transverse or longitudinal projector onto each argument, and a division by $\sum_{i=1}^{n}\Gamma_i^{(2)}(k_i^2)$, where $\Gamma_i^{(2)}(k^2) = \mathrm{T}(k^2)$ or $\Gamma_i^{(2)}(k^2) = a^{-1}\mathrm{L}(k^2)$. This gives all the $\Gamma^{(n)}$ uniquely in terms of $Q^{(1)}$ to $Q^{(n-1)}$.

Appendix E. Evaluation of $\Gamma^{(3)}_{\mathrm{gt}}$

We evaluate $\Gamma^{(3)}_{\mathrm{gt}}$, using the formulas of sec. 7, with the substitutions (9.14). From (7.15) and (8.6), we obtain

$$H^{(3)}_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = a^{-1}\tilde{\Gamma}^{(2)}_{\mu_1\mu}(k_1)\ \tilde{K}^{(2)}_{\mathrm{gt}\ \mu\mu_2\mu_3}(k_1; k_2, k_3) + (\text{cyclic})$$
$$= ia^{-1}g\left((k_3)_{\mu_3}\ [\ \tilde{\Gamma}^{(2)}_{\mu_1\mu_2}(k_1) - (1 \leftrightarrow 2)\ ] + (\text{cyclic})\right).$$

(E.1)

We apply transverse or longitudinal projectors to each Lorentz index, and use $\tilde{\Gamma}^{(2)}_{\lambda\mu}(k) = T(k^2)P^{\mathrm{T}}_{\lambda\mu}(k) + a^{-1}L(k^2)P^{\mathrm{L}}_{\lambda\mu}(k)$ to obtain

$$H^{(3)\mathrm{T\,T\,T}}_{\quad\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = 0,$$
$$H^{(3)\mathrm{T\,T\,L}}_{\quad\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = ia^{-1}g\,(T_1 - T_2)\,(k_3)_{\mu_3}\,[P^{\mathrm{T}}(k_1)P^{\mathrm{T}}(k_2)]_{\mu_1\mu_2},$$
$$H^{(3)\mathrm{T\,L\,L}}_{\quad\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = ia^{-1}g\left((a^{-1}L_3 - T_1)\,(k_2)_{\mu_2}\,[P^{\mathrm{T}}(k_1)P^{\mathrm{L}}(k_3)]_{\mu_1\mu_3} - (2 \leftrightarrow 3)\right)$$
$$H^{(3)\mathrm{L\,L\,L}}_{\quad\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = ia^{-2}g\left((L_2 - L_3)\,(k_1)_{\mu_1}\,[P^{\mathrm{L}}(k_2)P^{\mathrm{L}}(k_3)]_{\mu_2\mu_3} + \text{cyclic}\right),$$

(E.2)

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where we have used the notation $T_i \equiv T(k_i^2)$ and $L_i \equiv L(k_i^2)$. From (7.16) and (7.17), we obtain finally

$$\tilde{\Gamma}^{(3)\,\mathrm{T}\,\mathrm{T}\,\mathrm{T}}_{\mathrm{gt}\,\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = 0,$$

$$\tilde{\Gamma}^{(3)\,\mathrm{T}\,\mathrm{T}\,\mathrm{L}}_{\mathrm{gt}\,\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = -\,ig\,\frac{T_1 - T_2}{aT_1 + aT_2 + L_3}\,(k_3)_{\mu_3}\,[P^{\mathrm{T}}(k_1)P^{\mathrm{T}}(k_2)]_{\mu_1\mu_2},$$

$$\tilde{\Gamma}^{(3)\,\mathrm{T}\,\mathrm{L}\,\mathrm{L}}_{\mathrm{gt}\,\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = -\,ia^{-1}g\,\Big(\,\frac{L_3 - aT_1}{aT_1 + L_2 + L_3}\,(k_2)_{\mu_2}\,[P^{\mathrm{T}}(k_1)P^{\mathrm{L}}(k_3)]_{\mu_1\mu_3}$$

$$-\,\frac{L_2 - aT_1}{aT_1 + L_2 + L_3}\,(k_3)_{\mu_3}\,[P^{\mathrm{T}}(k_1)P^{\mathrm{L}}(k_2)]_{\mu_1\mu_2}\,\Big),$$

$$\tilde{\Gamma}^{(3)\,\mathrm{L}\,\mathrm{L}\,\mathrm{L}}_{\mathrm{gt}\,\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = -\,ia^{-1}g\,\Big(\,\frac{L_2 - L_3}{L_1 + L_2 + L_3}\,(k_1)_{\mu_1}\,[P^{\mathrm{L}}(k_2)P^{\mathrm{L}}(k_3)]_{\mu_2\mu_3} + (\mathrm{cyclic})\Big). \tag{E.3}$$

Appendix F. Evaluation of loop integrals

We evaluate the integral that appears in (11.4) namely

$$I_T \equiv \frac{1}{(k^2)^{\alpha_T + 2}\,(d-1)\,(2\pi)^d}\int d^d k_1\,\frac{k_1^2\,k^2 - (k_1 \cdot k)^2}{(k_1^2)^{1+\alpha_L}\,[(k - k_1)^2]^{1+\alpha_L}}. \tag{F.1}$$

We write this as

$$I_T = \frac{1}{(k^2)^{\alpha_T + 2}\,(d-1)\,\Gamma^2(1 + \alpha_L)}\int_0^\infty d\alpha d\beta\,\alpha^{\alpha_L}\,\beta^{\alpha_L}\,R_T, \tag{F.2}$$

where

$$R_T \equiv (2\pi)^{-d}\int d^d k_1\,[\,k_1^2\,k^2 - (k_1 \cdot k)^2\,]\,\exp[\,-\,\alpha k_1^2 - \beta(k_1 - k)^2\,]. \tag{F.3}$$

We complete the square in the exponent,

$$\alpha k_1^2 + \beta(k_1 - k)^2 = (\alpha + \beta)p^2 + (\alpha + \beta)^{-1}\alpha\beta k^2,$$

where $p = k_1 - (\alpha + \beta)^{-1}\beta k$, and obtain

$$R_T = \exp[\,-\,(\alpha + \beta)^{-1}\alpha\beta k^2\,]\,(2\pi)^{-d}\int d^d p\,[\,p^2\,k^2 - (p \cdot k)^2\,]\,\exp[\,-\,(\alpha + \beta)p^2\,]$$

$$= \frac{(d-1)\,k^2}{2\,(4\pi)^{d/2}\,(\alpha + \beta)^{1+d/2}}\,\exp[\,-\,(\alpha + \beta)^{-1}\alpha\beta k^2\,]. \tag{F.4}$$

This gives

$$I_T = \frac{1}{2\,(4\pi)^{d/2}(k^2)^{\alpha_T + 1}\,\Gamma^2(1 + \alpha_L)}\,S_T, \tag{F.5}$$

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where

$$S_T = \int_0^\infty d\alpha d\beta \, \frac{\alpha^{\alpha_L} \beta^{\alpha_L}}{(\alpha + \beta)^{1+d/2}} \, \exp[\, - (\alpha + \beta)^{-1} \alpha \beta k^2 \,]. \tag{F.6}$$

We insert the identity $1 = \int_0^\infty d\sigma \, \delta(\alpha + \beta - \sigma)$, and change variables according to $\alpha = \sigma\alpha'$ and $\beta = \sigma\beta'$. This gives, after dropping primes,

$$\begin{aligned}
S_T &= \int_0^\infty d\alpha d\beta d\sigma \, \delta(\alpha + \beta - 1) \, \alpha^{\alpha_L} \, \beta^{\alpha_L} \sigma^{2\alpha_L - d/2} \, \exp[\, - \alpha\beta\sigma k^2 \,] \\
&= (k^2)^{d/2 - 2\alpha_L - 1} \, \Gamma(2\alpha_L + 1 - d/2) \int_0^\infty d\alpha d\beta \, \delta(\alpha + \beta - 1) \, \alpha^{d/2 - \alpha_L - 1} \, \beta^{d/2 - \alpha_L - 1} \\
&= (k^2)^{d/2 - 2\alpha_L - 1} \, \frac{\Gamma(2\alpha_L + 1 - d/2) \, \Gamma^2(d/2 - \alpha_L)}{\Gamma(d - 2\alpha_L)}.
\end{aligned} \tag{F.7}$$

We obtain finally,

$$\begin{aligned}
I_T &= \frac{\Gamma(2\alpha_L + 1 - d/2) \, \Gamma^2(d/2 - \alpha_L)}{2 \, (4\pi)^{d/2} \, \Gamma^2(1 + \alpha_L) \, \Gamma(d - 2\alpha_L)} \\
&= \frac{\Gamma(2\alpha_L - 1) \, \Gamma^2(2 - \alpha_L)}{2 \, (4\pi)^2 \, \Gamma^2(1 + \alpha_L) \, \Gamma(4 - 2\alpha_L)},
\end{aligned} \tag{F.8}$$

for $d = 4$.

We also evaluate the integral that appears in (11.9),

$$I_L \equiv \frac{-2}{(k^2)^{1+\alpha_L} \, (2\pi)^d} \int d^d k_1 \, \frac{k^2 k_1^2 - (k \cdot k_1)^2}{(k_1^2)^{2+\alpha_T} \, [(k_1 - k)^2]^{\alpha_L} [(k_1 - k)^2 + k^2]}. \tag{F.9}$$

It contains the denominator $[(k_1 - k)^2 + k^2]$ that comes from the non-local vertex. This integral is convergent in the ultraviolet for $d < 4 + 2\alpha_T + 2\alpha_L$. We shall evaluate it for d satisfying this condition, and then continue in d. We write it as

$$I_L = \frac{-2}{(k^2)^{1+\alpha_L} \Gamma(2 + \alpha_T) \, \Gamma(\alpha_L)} \int_0^\infty d\alpha d\beta d\gamma \, \alpha^{1+\alpha_T} \, \beta^{\alpha_L - 1} \exp(-\gamma k^2) \, R_L, \tag{F.10}$$

where

$$R_L \equiv (2\pi)^{-d} \int d^d k_1 \, [\, k_1^2 \, k^2 - (k_1 \cdot k)^2 \,] \, \exp[\, - \alpha k_1^2 - (\beta + \gamma)(k_1 - k)^2 \,]. \tag{F.11}$$

We complete the square in the exponent,

$$\alpha k_1^2 + (\beta + \gamma)(k_1 - k)^2 = (\alpha + \beta + \gamma)p^2 + (\alpha + \beta + \gamma)^{-1}\alpha(\beta + \gamma)k^2,$$

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where $p = k_1 - (\alpha + \beta + \gamma)^{-1}(\beta + \gamma)k$, and obtain

$$
\begin{aligned}
R_L &= \exp\left[- (\alpha + \beta + \gamma)^{-1}\alpha(\beta + \gamma)\ k^2\ \right] \\
&\qquad \times (2\pi)^{-d} \int d^d p\ [\ p^2\ k^2 - (p \cdot k)^2\ ]\ \exp\left[- (\alpha + \beta + \gamma)p^2\ \right] \\
&= \frac{(d-1)\ k^2}{2\ (4\pi)^{d/2}\ (\alpha + \beta + \gamma)^{1+d/2}}\ \exp\left[- (\alpha + \beta + \gamma)^{-1}\alpha(\beta + \gamma)k^2\ \right].
\end{aligned}
\tag{F.12}
$$

This gives

$$
I_L = - \frac{(d-1)}{(k^2)^{\alpha_L}\ (4\pi)^{d/2}\ \Gamma(2+\alpha_T)\Gamma(\alpha_L)}\ S_L,
\tag{F.13}
$$

where

$$
S_L = \int_0^\infty d\alpha d\beta d\gamma\ \frac{\alpha^{1+\alpha_T}\ \beta^{\alpha_L - 1}}{(\alpha + \beta + \gamma)^{1+d/2}}\ \exp\left[- \gamma k^2 - (\alpha + \beta + \gamma)^{-1}\alpha(\beta + \gamma)k^2\ \right].
\tag{F.14}
$$

We insert the identity $1 = \int_0^\infty d\sigma\ \delta(\alpha + \beta + \gamma - \sigma)$, and change variables according to $\alpha = \sigma\alpha'$, $\beta = \sigma\beta'$ and $\gamma = \sigma\gamma'$. This gives, after dropping primes,

$$
\begin{aligned}
S_L &= \int_0^\infty d\alpha d\beta d\gamma d\sigma\ \delta(\alpha + \beta + \gamma - 1)\ \alpha^{1+\alpha_T}\ \beta^{\alpha_L - 1}\sigma^{1+\alpha_T+\alpha_L-d/2} \\
&\qquad \times\ \exp\{\ -\sigma[\gamma + \alpha(\beta + \gamma)]k^2\} \\
&= (k^2)^{-\alpha_T-\alpha_L-2+d/2}\ \Gamma(2 + \alpha_T + \alpha_L - d/2)\ M_L,
\end{aligned}
\tag{F.15}
$$

The argument of the Γ-function is positive in the region of convergence of the integral, $d < 4 + 2\alpha_T + 2\alpha_L$. Here M_L is the finite integral

$$
\begin{aligned}
M_L &\equiv \int_0^\infty d\alpha d\beta d\gamma\ \delta(\alpha + \beta + \gamma - 1)\ \alpha^{1+\alpha_T}\ \beta^{\alpha_L - 1}[\alpha(\beta + \gamma) + \gamma]^{\alpha_L} \\
&= \int_0^1 d\beta \int_0^{1-\beta} d\alpha\ \alpha^{1+\alpha_T}\ \beta^{\alpha_L - 1}(1 - \alpha^2 - \beta)^{\alpha_L},
\end{aligned}
\tag{F.16}
$$

where we have used $\alpha_T + 2\alpha_L = - (4 - d)/2$. This gives

$$
\begin{aligned}
I_L &= - \frac{(d-1)\ \Gamma(-\alpha_L)}{(4\pi)^{d/2}\ \Gamma(2+\alpha_T)\Gamma(\alpha_L)}\ M_L \\
&= \frac{(d-1)\ \Gamma(1-\alpha_L)}{(4\pi)^{d/2}\ \Gamma(2+\alpha_T)\Gamma(1+\alpha_L)}\ M_L.
\end{aligned}
\tag{F.17}
$$

Note that I_L is negative in the region of convergence of the integral, but after the continuation in d, it is positive.

To evaluate M_L we change variable to $x = \alpha^2$, and obtain

$$M_L = (1/2) \int_0^1 d\beta \int_0^{(1-\beta)^2} dx \; x^{\alpha_T/2} \; \beta^{\alpha_L-1}(1-\beta-x)^{\alpha_L}, \tag{F.18}$$

and upon changing variables to $x = (1-\beta)y$, we get

$$M_L = (1/2) \int_0^1 d\beta \int_0^{1-\beta} dy \; y^{\alpha_T/2} \; \beta^{\alpha_L-1}(1-\beta)^{1+\alpha_L+\alpha_T/2}(1-y)^{\alpha_L}. \tag{F.19}$$

We again use $\alpha_T + 2\alpha_L = -(4-d)/2$ to write this as

$$M_L = (1/2) \int_0^1 dy \; y^{\alpha_T/2}(1-y)^{\alpha_L} \int_0^{1-y} d\beta \; \beta^{\alpha_L-1}(1-\beta)^{d/4}. \tag{F.20}$$

This is integrable by quadrature for $d = 4$, and in this case it gives

$$
\begin{aligned}
M_L &= (1/2) \int_0^1 dy \; y^{-\alpha_L}\left(\frac{(1-y)^{2\alpha_L}}{\alpha_L} - \frac{(1-y)^{2\alpha_L+1}}{\alpha_L+1}\right) \\
&= (1/2)\left(\frac{\Gamma(1-\alpha_L)\,\Gamma(1+2\alpha_L)}{\alpha_L\,\Gamma(2+\alpha_L)} - \frac{\Gamma(1-\alpha_L)\,\Gamma(2+2\alpha_L)}{(\alpha_L+1)\,\Gamma(3+\alpha_L)}\right) \\
&= \frac{(-\alpha_L^2 + 2\alpha_L + 2)\,\Gamma(1-\alpha_L)\,\Gamma(2\alpha_L+1)}{2\alpha_L(\alpha_L+1)\,\Gamma(\alpha_L+3)},
\end{aligned}
\tag{F.21}
$$

where we used $\alpha_T = -2\alpha_L$. This gives finally

$$I_L = \frac{3\,(-\alpha_L^2 + 2\alpha_L + 2)\,\Gamma^2(1-\alpha_L)\,\Gamma(2\alpha_L+1)}{2\,(4\pi)^2\,\alpha_L\,\Gamma(2-2\alpha_L)\Gamma(\alpha_L+2)\,\Gamma(\alpha_L+3)}. \tag{F.22}$$

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Figure Captions

Fig. 1.   Fig. 1. Diagrammatic representation of the functional DS equation for the quantum effective drift force, $\mathcal{Q}(A)$, in the presence of external sources A, eq. (6.1). The vertices are the tree-level vertices of the drift force K. The internal lines represent the exact gluon propagator $\mathcal{D}(A)$ in the presence of the external source. The circle is the exact 3-gluon vertex of the quantum effective action $\Gamma(A)$ in the presence of the external source.

Fig. 2.   Fig. 2. Diagrammatic representation of the DS equation for the gluon propagator, eq. (8.1). The vertices are the tree level vertices of the drift force K. The internal lines are the exact gluon propagator D with sources set to 0. The circles represent the exact 3 and 4-gluon vertices of the quantum effective action Γ, with sources set to 0.
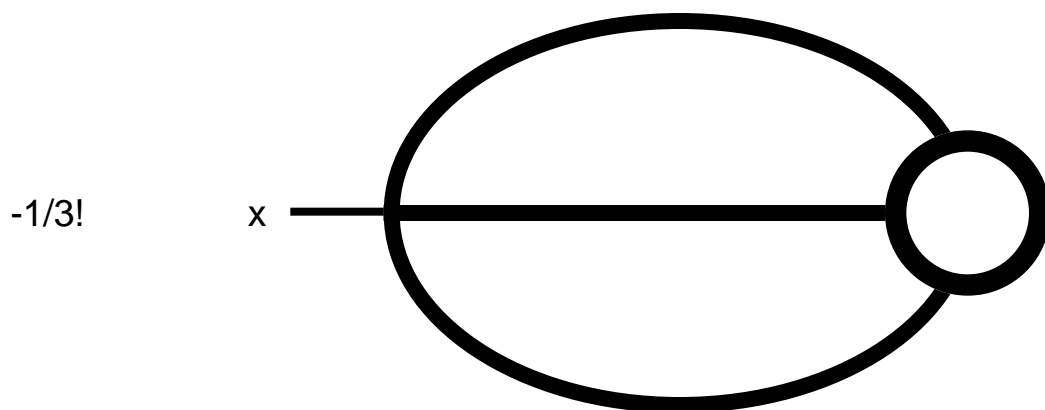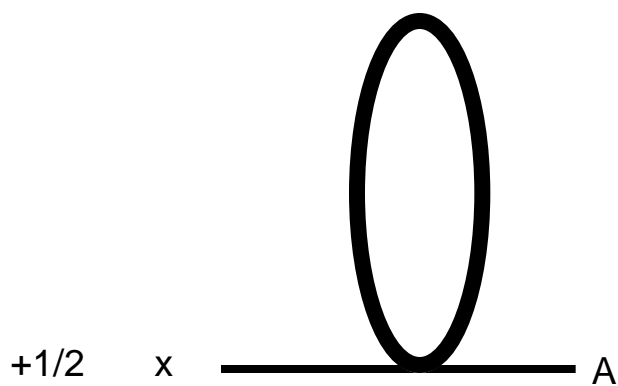
Figure 1

$Q(x) =$ ———— $K(x)$

$+ 1/2$ x

$+1/2$ x ———— A

$-1/3!$ x

Figure 2

$Q^{(1)}(x; y) =$ x ———————————— y

$- 1/2$ x ⬯ y

$+1/2$ x ⬯ y

$-1/3!$ x ⬯ y

$+1/2$ x ⬯ y